ON GENERALIZING THE AMG FRAMEWORK

ROBERT D. FALGOUT* AND PANAYOT S. VASSILEVSKI*

Abstract. We present a theory for algebraic multigrid (AMG) methods that allows for general smoothing processes and general coarsening approaches. The goal of the theory is to provide guidance in the development of new, more robust, AMG algorithms. In particular, we introduce several compatible relaxation methods and give theoretical justification for their use as tools for measuring the quality of coarse grids.

Key words. algebraic multigrid, compatible relaxation

1. Introduction. The algebraic multigrid (AMG) method was originally developed to solve general matrix equations using multigrid principles [7, 3, 8, 19, 4, 20]. The fact that it used only information in the underlying matrix made it attractive as a potential black box solver, a notion that has since been all but abandoned. Instead, a wide variety of AMG algorithms have been developed that target different problem classes and have different robustness and efficiency properties.

In recent years, much work has been done to increase the robustness of algebraic multigrid methods. The classical AMG method of Ruge and Stüben [20] was built upon heuristics based on properties of M-matrices. Although this algorithm works remarkably well for a wide variety of problems [12], the M-matrix assumption still limits its applicability. To address this, a new class of algorithms was developed based on multigrid theory: AMGe [9, 16], element-free AMGe [14], and spectral AMGe [11]. All of these algorithms (including Ruge-Stüben AMG) assume a basic framework in their construction: they assume that relaxation is a simple pointwise method, then they build the coarse-grid correction step to eliminate the so-called *algebraically smooth* error left over by the relaxation process. In the AMGe methods, this is done with the help of a *measure* and an associated approximation property that, if satisfied, implies uniform multigrid convergence. The approximation property induces a new heuristic that relates the accuracy of interpolation to the spectrum of the system matrix: namely, that eigenmodes with small associated eigenvalue must be interpolated well.

In this paper, we present a theory that generalizes the AMG framework to address even broader classes of problems. For example, the eddy current formulation of Maxwell's Equations (when discretized using the common Nédélec finite elements) has a particularly large (near) null-space. In the previous framework, it would be necessary to take all $O(N)$ of the null-space components to the coarse grid, yielding a non-optimal method. This difficulty can be overcome by using non-pointwise smoothers that damp some of the null-space components on the fine grid. Examples include overlapping block relaxation [1] and a form of Brandt's distributive relaxation [6, 21] described by Hiptmair in [15].

The theory presented here allows for more general smoothing processes, and changes the above AMGe heuristic in a subtle but important way. It also allows for general coarsening approaches, including vertex-based, cell-based, and agglomeration-based. Yet another aspect of the new theory and framework is *compatible relaxation*,

*Center for Applied Scientific Computing, Lawrence Livermore National Laboratory, P.O. Box 808, L-561, Livermore, CA 94551. E-mail: rfalgout@llnl.gov, panayot@llnl.gov.

†This work was performed under the auspices of the U. S. Department of Energy by University of California Lawrence Livermore National Laboratory under contract W-7405-Eng-48.

an idea originally proposed by Brandt [5]. We introduce several variants of compatible relaxation and give theoretical justification for its use. The hope is that this work will provide guidance in the development of new AMG methods able to handle difficult problems such as Maxwell's equations.

We assume that the reader is somewhat familiar with AMG research, as numerous comparisons will be made to AMGe and other methods such as smoothed aggregation [22]. In Section 2, we introduce two new measures and provide two-level convergence theory. In Section 3, we analyze the min-max problem for the new measures. In Section 4, we discuss the process of building interpolation, and provide additional theory to support this approach. In Section 5, we show how to use compatible relaxation to evaluate the measure and select coarse grids. In Section 6, we present two examples illustrating the application of the theory to real problems.

2. New Measures and Convergence Theory. We begin with some notation. Capital italic Roman letters (A, M, P, R) denote matrices and bold lowercase Roman and Greek letters denote vectors $(\mathbf{u}, \mathbf{v}, \boldsymbol{\varepsilon})$. Other lowercase letters denote scalars, while capital calligraphic letters denote sets and spaces $(\mathcal{C}, \mathcal{F}, \mathcal{S})$. We represent the standard Euclidean inner product by $\langle \cdot,\ \cdot \rangle$ with associated norm, $\|\cdot\| := \langle \cdot,\ \cdot \rangle^{1/2}$. The A-norm (also called the energy norm) is defined by $\|\cdot\|_A := \langle A\ \cdot,\ \cdot \rangle^{1/2}$.

Consider solving via algebraic multigrid the linear system

$$A\mathbf{u} = \mathbf{f}, \tag{2.1}$$

where A is a real symmetric positive definite (SPD) matrix, with $\mathbf{u}, \mathbf{f} \in \mathbb{R}^n$. We consider smoothers (relaxation methods) of the form,

$$\mathbf{u}_{k+1} = \mathbf{u}_k + M^{-1}\mathbf{r}_k, \tag{2.2}$$

where $\mathbf{r}_k = \mathbf{f} - A\mathbf{u}_k$ is the residual at the k^{th} iteration. The error propagation for this iteration is given by

$$\mathbf{e}_{k+1} = (I - M^{-1}A)\mathbf{e}_k. \tag{2.3}$$

We also assume that $(M + M^T - A)$ is SPD. It is easy to see that this is a necessary and sufficient condition for convergence (e.g., see the first line in the proof of Theorem 2.2), and hence a reasonable assumption.

Let $P : \mathbb{R}^{n_c} \to \mathbb{R}^n$ be the *interpolation* (or *prolongation*) operator, where $\mathbb{R}^{n_c}$ is a lower-dimensional (*coarse*) vector space, and define $Q : \mathbb{R}^n \to \mathbb{R}^n$ to be a projection onto range(P):

$$Q = PR, \tag{2.4}$$

for some restriction operator $R : \mathbb{R}^n \to \mathbb{R}^{n_c}$ such that $RP = I_c$, the identity on $\mathbb{R}^{n_c}$. Note that R is not the multigrid restriction operator (we will use P^T and the Galerkin procedure). Also note that the form of R will be important in the remaining sections of the paper.

Define the following measure (we will introduce a second, simpler measure later),

$$\mu(Q, \mathbf{e}) := \frac{\langle M(M + M^T - A)^{-1}M^T(I - Q)\mathbf{e},\ (I - Q)\mathbf{e}\rangle}{\langle A\mathbf{e},\ \mathbf{e}\rangle}. \tag{2.5}$$

This measure differs from the AMGe measure in [9] by the inclusion of the term $M(M + M^T - A)^{-1}M^T$ in the numerator. The additional term takes into account the general relaxation process in (2.2). It also provides a natural scaling that eliminates the need to pre-scale A to have diagonal equal one, as in the theory for AMGe.

We now prove that if the measure in (2.5) is bounded by a constant, then two-level multigrid converges uniformly. Furthermore, a smaller measure yields faster convergence. Denote the A-orthogonal projector onto range(P) by

$$Q_A := P(P^TAP)^{-1}P^TA, \tag{2.6}$$

so that $I - Q_A$ represents the error propagation matrix for the coarse-grid correction step. We first prove the following lemma.

LEMMA 2.1. *Let Q be any projection onto* range(P). *Assume that the following approximation property is satisfied for some constant K:*

$$\mu(Q, \mathbf{e}) \le K \quad \forall \mathbf{e} \in \mathbb{R}^n \setminus \{0\}. \tag{2.7}$$

If $\mathbf{e} \neq 0$ is A-orthogonal to range(P), *then*

$$\left\| (M + M^T - A)^{1/2} M^{-1} A\mathbf{e} \right\|^2 \ge \frac{1}{K} \langle A\mathbf{e},\ \mathbf{e} \rangle . \tag{2.8}$$

Proof. Note that range(Q) = range(P), hence, if $\mathbf{e}$ is A-orthogonal to range(P), then

$$\langle A\mathbf{e},\ Q\mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in \mathbb{R}^n. \tag{2.9}$$

Assume that (2.7) holds. From (2.9) and the Cauchy-Schwartz inequality, we have

$$\begin{aligned}\langle A\mathbf{e},\ \mathbf{e} \rangle &= \langle A\mathbf{e},\ (I - Q)\mathbf{e} \rangle \\ &= \left\langle (M + M^T - A)^{1/2} M^{-1} A\mathbf{e},\ (M + M^T - A)^{-1/2} M^T (I - Q)\mathbf{e} \right\rangle \\ &\le \left\| (M + M^T - A)^{1/2} M^{-1} A\mathbf{e} \right\| \left\| (M + M^T - A)^{-1/2} M^T (I - Q)\mathbf{e} \right\| \\ &\le \left\| (M + M^T - A)^{1/2} M^{-1} A\mathbf{e} \right\| K^{1/2} \langle A\mathbf{e},\ \mathbf{e} \rangle^{1/2} .\end{aligned}$$

The result (2.8) now follows by dividing through by $\langle A\mathbf{e},\ \mathbf{e} \rangle K^{1/2}$ and squaring the result. ☐

THEOREM 2.2. *Assume that approximation property* (2.7) *is satisfied for some constant K. Then $K \ge 1$ and*

$$\left\| (I - M^{-1}A)(I - Q_A)\mathbf{e} \right\|_A \le \left(1 - \frac{1}{K} \right)^{1/2} \|\mathbf{e}\|_A . \tag{2.10}$$

Proof. We have the following identity

$$\begin{aligned}\left\| (I - M^{-1}A)\mathbf{e} \right\|_A^2 = \langle A\mathbf{e},\ \mathbf{e} \rangle &- \left\langle A\mathbf{e},\ M^{-1}A\mathbf{e} \right\rangle \\ &- \left\langle M^{-1}A\mathbf{e},\ A\mathbf{e} \right\rangle + \left\langle AM^{-1}A\mathbf{e},\ M^{-1}A\mathbf{e} \right\rangle \\ = \langle A\mathbf{e},\ \mathbf{e} \rangle &- \left\langle (M + M^T - A)(M^{-1}A)\mathbf{e},\ (M^{-1}A)\mathbf{e} \right\rangle .\end{aligned}$$

Replacing $\mathbf{e}$ with $(I - Q_A)\mathbf{e}$ and applying the result in Lemma 2.1 yields

$$\begin{aligned}\left\| (I - M^{-1}A)(I - Q_A)\mathbf{e} \right\|_A^2 &\le \left(1 - \frac{1}{K} \right) \|(I - Q_A)\mathbf{e}\|_A^2 \\ &\le \left(1 - \frac{1}{K} \right) \|\mathbf{e}\|_A^2 .\end{aligned}$$

To show that $K \geq 1$, note that the identity at the beginning of the proof implies (since norms are non-negative)

$$\left\|(M + M^T - A)^{1/2} M^{-1} A\mathbf{e}\right\|^2 \leq \langle A\mathbf{e},\ \mathbf{e}\rangle .$$

The result follows by restricting $\mathbf{e} \neq 0$ to be A-orthogonal to range(P) and applying Lemma 2.1. ▯

The result in Theorem 2.2 is similar to the AMGe result in [9], but applies to more general relaxation methods (than Richardson relaxation). As in AMGe, the bound on the convergence factor approaches 1 as K becomes large, while a smaller K yields a smaller bound on the convergence factor. Note, however, that neither the new measure μ nor the corresponding convergence result reduces to the AMGe measure or convergence result in the case of Richardson relaxation. To complete the connection between the two theories, we now introduce a second, simpler measure:

$$\mu_\sigma(Q, \mathbf{e}) := \frac{\langle \sigma(M)(I - Q)\mathbf{e},\ (I - Q)\mathbf{e}\rangle}{\langle A\mathbf{e},\ \mathbf{e}\rangle}, \tag{2.11}$$

where $\sigma(M) := \frac{1}{2}(M + M^T)$ is the symmetric part of M. Note that the term $\sigma(M)$ can be replaced equivalently by M, but the symmetric form of this measure is more natural in the theory that follows. The relationship between the measures μ and μ_σ is given in the next lemma.

Lemma 2.3. *Assume that $(M + M^T - A)$ is SPD. Then,*

$$\mu(Q, \mathbf{e}) \leq \frac{\Delta^2}{2 - \omega}\ \mu_\sigma(Q, \mathbf{e}), \tag{2.12}$$

where $\Delta \geq 1$ measures the deviation of M from its symmetric part in the sense that

$$\langle M\mathbf{v},\ \mathbf{w}\rangle \leq \Delta \langle \sigma(M)\mathbf{v},\ \mathbf{v}\rangle^{1/2} \langle \sigma(M)\mathbf{w},\ \mathbf{w}\rangle^{1/2}, \tag{2.13}$$

and where

$$0 < \omega := \lambda_{\max}(\sigma(M)^{-1}A) < 2. \tag{2.14}$$

Proof. Note that since $(M + M^T - A)$ is SPD, then both $\sigma(M)$ and $\sigma(M^{-1})$ are also SPD. From (2.13), letting $\mathbf{v} = M^{-1}\mathbf{x}$ and $\mathbf{w} = \sigma(M)^{-1}\mathbf{x}$, we have that

$$\left\langle \sigma(M)^{-1}\mathbf{x},\ \mathbf{x}\right\rangle^2 \leq \Delta^2 \left\langle \sigma(M)M^{-1}\mathbf{x},\ M^{-1}\mathbf{x}\right\rangle \left\langle \sigma(M)^{-1}\mathbf{x},\ \mathbf{x}\right\rangle .$$

Dividing both sides by $\left\langle \sigma(M)^{-1}\mathbf{x},\ \mathbf{x}\right\rangle$ yields

$$\begin{aligned}\left\langle \sigma(M)^{-1}\mathbf{x},\ \mathbf{x}\right\rangle &\leq \Delta^2 \left\langle MM^{-1}\mathbf{x},\ M^{-1}\mathbf{x}\right\rangle \\ &= \Delta^2 \left\langle M^{-1}\mathbf{x},\ \mathbf{x}\right\rangle \\ &= \Delta^2 \left\langle \sigma(M^{-1})\mathbf{x},\ \mathbf{x}\right\rangle .\end{aligned}$$

From this and (2.14), we then have

$$\begin{aligned}
\mu(Q,\mathbf{e}) &= \frac{\langle M(M+M^T-A)^{-1}M^T(I-Q)\mathbf{e},\ (I-Q)\mathbf{e}\rangle}{\langle A\mathbf{e},\ \mathbf{e}\rangle} \\
&\le \max_{\mathbf{x}} \frac{\langle M(M+M^T-A)^{-1}M^T\mathbf{x},\ \mathbf{x}\rangle}{\langle \sigma(M)\mathbf{x},\ \mathbf{x}\rangle}\ \mu_\sigma(Q,\mathbf{e}) \\
&\le \left(\min_{\mathbf{x}} \frac{\langle (M(M+M^T-A)^{-1}M^T)^{-1}\mathbf{x},\ \mathbf{x}\rangle}{\langle \sigma(M^{-1})\mathbf{x},\ \mathbf{x}\rangle}\right)^{-1} \Delta^2\ \mu_\sigma(Q,\mathbf{e}) \\
&= \frac{\Delta^2}{\lambda_{\min}(\sigma(M^{-1})^{-1}(2\sigma(M^{-1})-M^{-T}AM^{-1}))}\ \mu_\sigma(Q,\mathbf{e}) \\
&= \frac{\Delta^2}{2-\omega}\ \mu_\sigma(Q,\mathbf{e}).
\end{aligned}$$

□

Lemma 2.3 provides an obvious corollary to Theorem 2.2 for measure μ_σ. This corollary is the analogue to the AMGe two-level convergence theory in [9]. To see this, note that for a weighted Richardson iteration with weight ω_r, we have that $M^{-1} = \omega_r \|A\|^{-1} I$. If we assume that the AMGe measure is bounded by some constant K_r, then the lemma implies that $\Delta = 1$, $\omega = \omega_r$, and hence,

$$\mu(Q,\mathbf{e}) \le \omega_r^{-1}(2-\omega_r)^{-1}\,\|A\|\ \ K_r.$$

Applying Theorem 2.2 then yields the AMGe convergence result.

In order for μ_σ to be a useful measure in practice, we need the constants ω and Δ to be "good" constants. In particular, we want both constants to be mesh independent, and we want ω to be bounded away from two. Bounding ω away from two is always possible by using appropriate weighting factors in the relaxation method. In the classical setting, this requirement is equivalent to satisfying a smoothing property; in general, it means that the smoother must damp large eigenmodes of A. Note that this does not preclude the smoother from also damping small eigenmodes (e.g., as required for Maxwell's equations).

To further elaborate on the constants ω and Δ, consider the discrete Laplacian on a uniform grid. First, define $m := \lambda_{\max}(D^{-1}A)$, where D is the diagonal of A. For weighted Jacobi relaxation with weighting factor 2/3, we have that $\omega = (2/3)m$. Since $m \le 2$ for the Laplacian, then $\omega \le 4/3$. For Gauss-Seidel relaxation, let $A = D+L+L^T$, where L is the strictly lower-triangular part of A. Then, $M = D+L$ implies that $\sigma(M) = \frac{1}{2}(D+A)$, and hence,

$$\omega = \lambda_{\max}\left[2(D+A)^{-1}A\right] = \frac{2}{1+m^{-1}}. \tag{2.15}$$

For the Laplacian, this again implies that $\omega \le 4/3$. We can also use (2.15) to estimate ω in more general settings. For example, in the case of finite elements, one can show that m is not larger than the maximum number of element degrees of freedom. Likewise for any sparse matrix A, one can show that m is not larger than the maximum number of nonzeros per row (column) of A.

The constant Δ is equal to 1 when M is symmetric. As an example of a nonsymmetric M, again, consider Gauss-Seidel. With m equal to the maximum number of nonzeros in a row (column) of A, and letting $\mathbf{v} = (v_i)$, $\mathbf{w} = (w_i)$, and $A = (a_{ij})$, we have,

$$\begin{aligned}
\langle M\mathbf{v},\ \mathbf{w}\rangle &\leq \sum_{j\leq i: a_{ij}\neq 0} |a_{ij}||v_j||w_i| \\
&\leq \sum_{j\leq i: a_{ij}\neq 0} \sqrt{a_{jj}}\sqrt{a_{ii}}|v_j||w_i| \\
&\leq \left[\sum_{j\leq i: a_{ij}\neq 0} a_{jj}(v_j)^2\right]^{1/2} \left[\sum_{j\leq i: a_{ij}\neq 0} a_{ii}(w_i)^2\right]^{1/2} \\
&\leq 1/2\,(m+1)\ \langle D\mathbf{v},\ \mathbf{v}\rangle^{1/2}\ \langle D\mathbf{w},\ \mathbf{w}\rangle^{1/2} \\
&\leq 1/2\,(m+1)\ \langle (D+A)\mathbf{v},\ \mathbf{v}\rangle^{1/2}\ \langle (D+A)\mathbf{w},\ \mathbf{w}\rangle^{1/2} \\
&= (m+1)\ \langle \sigma(M)\mathbf{v},\ \mathbf{v}\rangle^{1/2}\ \langle \sigma(M)\mathbf{w},\ \mathbf{w}\rangle^{1/2}.
\end{aligned}$$

3. The Min-Max Problem. In this section, we analyze the optimal min-max solution of the measures (2.5) and (2.11), and use the results as a discussion point for relating and comparing the new theory to existing methods such as AMGe, spectral AMGe, and smoothed aggregation [22]. We also introduce generalized notions of the C-pt (coarse points) and F-pt (fine points) terminology used in the classical Ruge-Stüben AMG algorithm. The material in this section serves as a launching pad for the ideas and results in the remainder of the paper.

To analyze the min-max solution of (2.5) and (2.11), we analyze the following base measure:

$$\mu_X(Q,\mathbf{e}) := \frac{\langle X(I-Q)\mathbf{e},\ (I-Q)\mathbf{e}\rangle}{\langle A\mathbf{e},\ \mathbf{e}\rangle}, \tag{3.1}$$

where, here again, Q has the form $Q = PR$ for some restriction operator $R : \mathbb{R}^n \rightarrow \mathbb{R}^{n_c}$ such that $RP = I_c$, and where X represents any given SPD matrix. In the remainder of the paper, it will be important that we fix R so that it does not depend on P (as in spectral AMGe). This operator defines the *coarse-grid variables* [5], $\mathbf{u}_c = R\mathbf{u}$, and specifies, for example, whether they are a subset of the fine-grid variables (vertex-centered), averages of fine-grid variables (cell-centered), or coefficients of fine-grid basis functions (agglomeration, e.g., as in spectral AMGe or smoothed aggregation). The coarse-grid variables, $R\mathbf{u}$, are analogous to C-pts in Ruge-Stüben AMG.

Now, define $S : \mathbb{R}^{n_s} \rightarrow \mathbb{R}^n$, where $n_s = n - n_c$, such that $RS = 0$. Think of range(S) as the "smoother space", i.e., the space on which the smoother must be effective. Note that S is not unique (but range(S) is). The variables, $S^T\mathbf{u}$, are analogous to F-pts. Note also that S and R^T define an orthogonal decomposition of $\mathbb{R}^n$. That is, any vector $\mathbf{e}$ can be written as $\mathbf{e} = S\mathbf{e}_s + R^T\mathbf{e}_c$, for some $\mathbf{e}_s$ and $\mathbf{e}_c$. We will see in Theorem 3.1 below that the min-max problem of this section also induces an A-orthogonal decomposition of $\mathbb{R}^n$ involving the operator S.

THEOREM 3.1. *Assume we are given a coarse grid Ω_c, and define*

$$\mu_X^\star := \min_P \max_{\mathbf{e}\neq 0} \mu_X(PR,\mathbf{e}). \tag{3.2}$$

The arg min *of* (3.2), $P_\star$, *satisfies*

$$P_\star^T A S = 0. \tag{3.3}$$

The minimum is given by

$$\mu_X^\star = \frac{1}{\lambda_{\min}((S^TXS)^{-1}(S^TAS))}. \tag{3.4}$$

Proof. Note that since $Q = PR$, $RP = I_c$, and $RS = 0$, we have

$$(I - Q)P = 0; \qquad (I - Q)S = S. \tag{3.5}$$

Also note that $\mathbf{e} - PR\mathbf{e} = (I-Q)\mathbf{e} \in \text{range}(S)$ since $R(I-Q) = 0$. Hence $\mathbf{e} = S\mathbf{e}_s + P\mathbf{e}_c$ for some $\mathbf{e}_s$ and $\mathbf{e}_c = R\mathbf{e}$. From (3.2), using (3.5), we then have that

$$\mu_X^\star = \min_P \max_{\mathbf{e}_c, \mathbf{e}_s} \frac{\langle XS\mathbf{e}_s,\ S\mathbf{e}_s\rangle}{\langle AS\mathbf{e}_s,\ S\mathbf{e}_s\rangle + 2\langle AS\mathbf{e}_s,\ P\mathbf{e}_c\rangle + \langle AP\mathbf{e}_c,\ P\mathbf{e}_c\rangle} \tag{3.6}$$

$$= \min_P \max_{\mathbf{e}_s} \frac{\langle XS\mathbf{e}_s,\ S\mathbf{e}_s\rangle}{\min_{\mathbf{e}_c}\left(\langle S^TAS\mathbf{e}_s,\ \mathbf{e}_s\rangle + 2\langle P^TAS\mathbf{e}_s,\ \mathbf{e}_c\rangle + \langle P^TAP\mathbf{e}_c,\ \mathbf{e}_c\rangle\right)}. \tag{3.7}$$

The denominator in (3.7) is a quadratic form in the variable $\mathbf{e}_c$ with solution

$$\mathbf{e}_c = -(P^TAP)^{-1}P^TAS\mathbf{e}_s. \tag{3.8}$$

Plugging (3.8) back into (3.7) gives

$$\mu_X^\star = \min_P \max_{\mathbf{e}_s \neq 0} \frac{\langle XS\mathbf{e}_s,\ S\mathbf{e}_s\rangle}{\langle S^TAS\mathbf{e}_s,\ \mathbf{e}_s\rangle - \langle (P^TAP)^{-1}P^TAS\mathbf{e}_s,\ P^TAS\mathbf{e}_s\rangle}. \tag{3.9}$$

Since the second term in the denominator of (3.9) is non-negative for any $\mathbf{e}_s$, the arg min must satisfy $P_\star^TAS = 0$. Hence,

$$\mu_X^\star = \max_{\mathbf{e}_s \neq 0} \frac{\langle S^TXS\mathbf{e}_s,\ \mathbf{e}_s\rangle}{\langle S^TAS\mathbf{e}_s,\ \mathbf{e}_s\rangle} = \frac{1}{\lambda_{\min}((S^TXS)^{-1}(S^TAS))}.$$

□

Theorem 3.1 is used to motivate the main result in Section 4. It will also be used to prove many of the results in Sections 4 and 5. An interesting corollary to the theorem is the following.

Corollary 3.2. *The optimal $P_\star$ in Theorem 3.1 is given by the formula:*

$$P_\star = \begin{bmatrix} S & R^T \end{bmatrix} \begin{bmatrix} -(S^TAS)^{-1}(S^TAR^T) \\ I \end{bmatrix} = (I - S(S^TAS)^{-1}S^TA)R^T. \tag{3.10}$$

Proof. This is obtained by solving the equation $S^TAP_\star = 0$. For any $\mathbf{v}$ consider $\mathbf{w} = P_\star\mathbf{v}_c$ and use its decomposition $\mathbf{w} = S\mathbf{w}_s + R^T\mathbf{w}_c$. We have, $\mathbf{v}_c = RP_*\mathbf{v}_c = R\mathbf{w} = RR^T\mathbf{w}_c = \mathbf{w}_c$. On the other hand, since $S^TA\mathbf{w} = 0$ one arrives at

$$S^TAS\mathbf{w}_s + S^TAR^T\mathbf{w}_c = 0.$$

That is, $\mathbf{w}_s = -(S^TAS)^{-1}S^TAR^T\mathbf{w}_c = -(S^TAS)^{-1}S^TAR^T\mathbf{v}_c$. Thus

$$P_\star\mathbf{v}_c = \left(-S(S^TAS)^{-1}S^TA + I\right)R^T\mathbf{v}_c,$$

which completes the proof. □

REMARK 3.1. *The first expression in* (3.10) *can be viewed as a generalization of the optimal interpolation for the AMGe measure (see Corollary 3.3 below). Alternatively, the second expression in* (3.10) *can be viewed as a kind of smoothed aggregation method. That is, the operator* R^T *is a type of tentative prolongator, and the term* $(I - S(S^TAS)^{-1}S^TA)$ *is a type of smoother (because it removes error components in the "smoother space" spanned by* S*). The interpolation operator in the smoothed aggregation method is formed similarly by smoothing a tentative prolongation operator, except that a simpler, local smoother is used. Another similarity is that the smoothed aggregation smoother is designed to leave unchanged the kernel components in* $\text{range}(R^T)$ *(those kernel components that are representable on the coarse grid). In* (3.10)*, the fact that* $\text{range}(S)$ *is* A*-orthogonal to* $\text{range}(P_\star)$ *also insures this.*

The following two corollaries specialize the results in Theorem 3.1 and Corollary 3.2 to the particular cases of AMGe and spectral AMGe. These results are useful primarily because of the insight and guidance they provide for developing algorithms in these settings.

COROLLARY 3.3. *Assume that* P *and* R *are as in AMGe and have the specific forms*

$$P = \begin{bmatrix} W \\ I \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & I \end{bmatrix}, \tag{3.11}$$

where we have reordered the equations so that

$$A = \begin{bmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{bmatrix}. \tag{3.12}$$

Let $X = \|A\|I$ *in* (3.1)*. Then, the* $\arg\min$ *and minimum of* (3.2) *are given by*

$$P_\star = \begin{bmatrix} -A_{ff}^{-1}A_{fc} \\ I \end{bmatrix}, \qquad \mu_X^\star = \frac{\|A\|}{\lambda_{\min}(A_{ff})}. \tag{3.13}$$

Proof. Let $S = [\ I \quad 0\ ]^T$. Then $RS = 0$ and $S^TAS = A_{ff}$. The result then follows trivially from (3.10) and (3.4). ▯

COROLLARY 3.4. *Assume that* R *has the form*

$$R^T = [\mathbf{p}_1, ..., \mathbf{p}_c], \tag{3.14}$$

where the $\mathbf{p}_i$*,* $1 \leq i \leq n$*, are the orthonormal eigenvectors of* A *with corresponding eigenvalues* $\lambda_1 \leq \cdots \leq \lambda_c \leq \cdots \leq \lambda_n$*. Let* $X = \|A\|\, I$ *in* (3.1)*. Then, the* $\arg\min$ *and minimum of* (3.2) *are given by*

$$P_\star = R^T, \qquad \mu_X^\star = \frac{\|A\|}{\lambda_{c+1}} = \frac{\lambda_n}{\lambda_{c+1}}. \tag{3.15}$$

Proof. Let $S = [\mathbf{p}_{c+1}, ..., \mathbf{p}_n]$. Then $RS = 0$ and $S^TAS = \text{diag}(\lambda_{c+1}, ..., \lambda_n)$. The result then follows trivially from (3.10) and (3.4). ▯

Now, consider tailoring the base min-max problem (3.2) to the case of the new measures in (2.5) and (2.11). Assume again that Q has the form $Q = PR$ for some fixed restriction operator $R : \mathbb{R}^n \to \mathbb{R}^{n_c}$ such that $RP = I_c$. As before, define

$S : \mathbb{R}^{n_s} \to \mathbb{R}^n$ such that $RS = 0$, and assume we are given a coarse grid Ω_c. Define, based on (2.11) and (2.11),

$$\mu^\star := \min_P \max_{\mathbf{e}\neq 0} \mu(PR, \mathbf{e}) \tag{3.16}$$

$$\mu_\sigma^\star := \min_P \max_{\mathbf{e}\neq 0} \mu_\sigma(PR, \mathbf{e}). \tag{3.17}$$

The quantities $\mu^\star$ and $\mu_\sigma^\star$ measure the ability of the coarse grid to represent algebraically smooth error, where algebraically smooth error is defined to be error components that are not being effectively damped by the more general relaxation process in (2.2). Strictly speaking, this interpretation of $\mu^\star$ and $\mu_\sigma^\star$ assumes that the interpolation operator is the optimal one; i.e., that $P = P_\star$. Hence, given a coarse grid, small quantities indicate that there exists *some* interpolation operator that can interpolate smooth error. Whether or not there exists a practical (e.g., local) interpolation operator is not addressed in this paper. However, empirical evidence so far indicates that $\mu^\star$ and $\mu_\sigma^\star$ are useful measures in practice, particularly for PDE problems.

4. Building Interpolation. In the previous section, we defined the quantities $\mu^\star$ and $\mu_\sigma^\star$ as indicators of the ability of the coarse grid to represent smooth error. Assuming that either of these quantities is “small” (we will present an efficient approach for estimating $\mu^\star$ and $\mu_\sigma^\star$ in the next section), we then need to build an interpolation operator. In practice, this means that we must somehow localize the new measure. However, note that the result (3.3) in Theorem 3.1 does not depend on the X in (3.1). This suggests the possibility that, once an adequate coarse grid has been chosen, the procedure for building an interpolation operator can be done without knowledge of the relaxation process. This is quantified in the next lemma and theorem.

LEMMA 4.1. *The following statements are equivalent, where $Q = PR$, P, R and S are as before, and where $\eta \geq 1$ is some constant:*

$$\langle AQ\mathbf{e},\ Q\mathbf{e}\rangle \leq \eta\,\langle A\mathbf{e},\ \mathbf{e}\rangle\,,\ \textit{for all}\ \mathbf{e}; \tag{4.1}$$

$$\langle A(I-Q)\mathbf{e},\ (I-Q)\mathbf{e}\rangle \leq \eta\,\langle A\mathbf{e},\ \mathbf{e}\rangle\,,\ \textit{for all}\ \mathbf{e}; \tag{4.2}$$

$$\langle AP\mathbf{e}_c,\ S\mathbf{e}_s\rangle^2 \leq (1-\frac{1}{\eta})\,\langle AP\mathbf{e}_c,\ P\mathbf{e}_c\rangle\,\langle AS\mathbf{e}_s,\ S\mathbf{e}_s\rangle\,,\ \textit{for all}\ \mathbf{e}_c, \mathbf{e}_s. \tag{4.3}$$

Proof. We first show that the approximate harmonic property of P (4.1) implies the strengthened Cauchy-Schwarz inequality (4.3). Letting $\mathbf{e} = tS\mathbf{e}_s + P\mathbf{e}_c$ for any $\mathbf{e}_c$, $\mathbf{e}_s$ and any real t, and noting that $Q\mathbf{e} = P\mathbf{e}_c$, then (4.1) leads to the following quadratic inequality for t,

$$t^2\,\langle AS\mathbf{e}_s,\ S\mathbf{e}_s\rangle + 2t\,\langle AP\mathbf{e}_c,\ S\mathbf{e}_s\rangle + (1-\frac{1}{\eta})\,\langle AP\mathbf{e}_c,\ P\mathbf{e}_c\rangle \geq 0.$$

This implies that the discriminant of the above quadratic form is non-positive, which is exactly the strengthened Cauchy-Schwarz inequality (4.3). In the same way, we can also show that (4.2) implies (4.3) by noting that $(I-Q)\mathbf{e} = tS\mathbf{e}_s$.

To show that the strengthened Cauchy-Schwarz inequality (4.3) implies the approximate harmonic property (4.1), let $\mathbf{e} = S\mathbf{e}_s + R^T\mathbf{e}_c$ and note that $R(I-Q)\mathbf{e} = 0$. Therefore, there is a $\hat{\mathbf{e}}_s$ such that $(I-Q)\mathbf{e} = S\hat{\mathbf{e}}_s$. That is, $\mathbf{e} = S\hat{\mathbf{e}}_s + P\mathbf{e}_c$, and one has

$$\langle A\mathbf{e},\ \mathbf{e}\rangle = \langle AS\hat{\mathbf{e}}_s,\ S\hat{\mathbf{e}}_s\rangle + 2\,\langle AS\hat{\mathbf{e}}_s,\ P\mathbf{e}_c\rangle + \langle AP\mathbf{e}_c,\ P\mathbf{e}_c\rangle\,.$$

Using (4.3) implies

$$\begin{aligned}\langle A\mathbf{e},\ \mathbf{e}\rangle &\geq \langle AS\hat{\mathbf{e}}_s,\ S\hat{\mathbf{e}}_s\rangle - 2\sqrt{1-\frac{1}{\eta}}\,\langle AS\hat{\mathbf{e}}_s,\ S\hat{\mathbf{e}}_s\rangle^{1/2}\langle AP\mathbf{e}_c,\ P\mathbf{e}_c\rangle^{1/2} + \langle AP\mathbf{e}_c,\ P\mathbf{e}_c\rangle \\ &= \frac{1}{\eta}\langle AP\mathbf{e}_c,\ P\mathbf{e}_c\rangle + \left[\langle AS\hat{\mathbf{e}}_s,\ S\hat{\mathbf{e}}_s\rangle^{1/2} - \sqrt{1-\frac{1}{\eta}}\,\langle AP\mathbf{e}_c,\ P\mathbf{e}_c\rangle^{1/2}\right]^2 \\ &\geq \frac{1}{\eta}\langle AQ\mathbf{e},\ Q\mathbf{e}\rangle .\end{aligned}$$

In the same way, we can also show that (4.3) implies (4.2). ▯

Theorem 4.2. *Define μ_X and $\mu_X^\star$ as in* (3.1) *and* (3.2)*, for any SPD matrix X. Assume that a coarse grid has been chosen, and that an interpolation operator P has been defined, such that the following conditions hold:*

C1: *$\mu_X^\star \leq K$, for some constant K;*

C2: (4.1)*,* (4.2)*, or* (4.3) *holds for some constant $\eta \geq 1$.*

Then, the following weak approximation property holds,

$$\mu_X(Q,\mathbf{e}) \leq \eta K \quad \forall \mathbf{e} \in \mathbb{R}^n \setminus \{0\}. \tag{4.4}$$

Proof. From Lemma 4.1, we can assume the strengthened Cauchy-Schwarz inequality (4.3). Now, consider the left-hand side of the desired inequality (4.4) and decompose $\mathbf{e} = S\mathbf{e}_s + R^T\mathbf{e}_c$. Note that $R(I-Q)\mathbf{e} = 0$, which implies there is a $\hat{\mathbf{e}}_s$ such that $(I-Q)\mathbf{e} = S\hat{\mathbf{e}}_s$. Hence, using (4.3) and Theorem 3.1, we have

$$\begin{aligned}\max_{\mathbf{e}} \mu_X(Q,\mathbf{e}) &= \max_{\hat{\mathbf{e}}_s}\max_{\mathbf{e}_c} \frac{\langle XS\hat{\mathbf{e}}_s,\ S\hat{\mathbf{e}}_s\rangle}{\langle A(S\hat{\mathbf{e}}_s + P\mathbf{e}_c),\ (S\hat{\mathbf{e}}_s + P\mathbf{e}_c)\rangle} \\ &= \max_{\hat{\mathbf{e}}_s}\max_{\mathbf{e}_c}\max_{t\in\mathbb{R}} \frac{\langle XS\hat{\mathbf{e}}_s,\ S\hat{\mathbf{e}}_s\rangle}{\langle A(S\hat{\mathbf{e}}_s + tP\mathbf{e}_c),\ (S\hat{\mathbf{e}}_s + tP\mathbf{e}_c)\rangle} \\ &= \max_{\hat{\mathbf{e}}_s}\max_{\mathbf{e}_c} \frac{\langle XS\hat{\mathbf{e}}_s,\ S\hat{\mathbf{e}}_s\rangle}{\min_{t\in\mathbb{R}}\langle A(S\hat{\mathbf{e}}_s + tP\mathbf{e}_c),\ (S\hat{\mathbf{e}}_s + tP\mathbf{e}_c)\rangle} \\ &\leq \max_{\hat{\mathbf{e}}_s}\max_{\mathbf{e}_c} \frac{\langle XS\hat{\mathbf{e}}_s,\ S\hat{\mathbf{e}}_s\rangle}{\langle AS\hat{\mathbf{e}}_s,\ S\hat{\mathbf{e}}_s\rangle - \frac{\langle AP\mathbf{e}_c,\ S\hat{\mathbf{e}}_s\rangle^2}{\langle AP\mathbf{e}_c,\ P\mathbf{e}_c\rangle}} \\ &\leq \max_{\hat{\mathbf{e}}_s}\max_{\mathbf{e}_c} \frac{\langle XS\hat{\mathbf{e}}_s,\ S\hat{\mathbf{e}}_s\rangle}{\langle AS\hat{\mathbf{e}}_s,\ S\hat{\mathbf{e}}_s\rangle - (1-\frac{1}{\eta})\langle AS\hat{\mathbf{e}}_s,\ S\hat{\mathbf{e}}_s\rangle} \\ &= \eta \max_{\hat{\mathbf{e}}_s} \frac{\langle XS\hat{\mathbf{e}}_s,\ S\hat{\mathbf{e}}_s\rangle}{\langle AS\hat{\mathbf{e}}_s,\ S\hat{\mathbf{e}}_s\rangle} \\ &= \eta\mu_X^\star \\ &\leq \eta K.\end{aligned}$$

▯

The corollaries to Theorem 4.2 for measures μ and μ_σ separate coarse-grid correction into two distinct parts: **C1** insures the quality of the coarse grid, i.e., its ability to represent algebraically smooth error components; and **C2** insures that these smooth components are adequately interpolated. Hence, once an adequate coarse grid is chosen, it is sufficient to build interpolation based on any one of the three statements in **C2**. In fact, the following result holds.

COROLLARY 4.3. *The statements in* **C2** *are necessary conditions for obtaining a uniformly convergent method.*

Proof. To see this in the case of measure μ_σ, note that our assumption that $(M+M^T-A)$ is SPD implies that $2\langle\sigma(M)\mathbf{e},\ \mathbf{e}\rangle \geq \langle A\mathbf{e},\ \mathbf{e}\rangle$. Hence, an approximation property that bounds measure μ_σ (with constant K_σ) also implies (4.2) (with $\eta = 2K_\sigma$). □

The significance of the above result is that the statements in **C2** nowhere involve the relaxation process. This implies that we can construct interpolation coefficients (again, assuming a coarse grid has already been chosen) using previously developed methods, even those methods that assumed a pointwise smoother. For example, in AMGe, a local procedure is used for constructing interpolation that produces an approximation property of the form

$$\|A\|\,\|(I-Q)\mathbf{e}\|^2 \leq \eta\,\langle A\mathbf{e},\ \mathbf{e}\rangle\,. \tag{4.5}$$

But, it is obvious that this also implies (4.2), which in turn implies the more general result in Theorem 4.2. Note that, even if the constant η is sharp in (4.5), this may be an extremely pessimistic constant for (4.2). See Section 6.1 for an example.

5. Compatible Relaxation. In this section, we introduce the idea of *compatible relaxation* and show how its convergence rate may be used to estimate the quantities $\mu^\star$ and $\mu_\sigma^\star$ in (3.16) and (3.17). That is, we will show how compatible relaxation may be used to insure **C1** of Theorem 4.2. We will present four variants of compatible relaxation, each having its own advantages and disadvantages, and suggest a simple algorithm for using these techniques to choose coarse grids in algebraic multigrid methods.

Compatible relaxation, as defined by Brandt [5], is *a modified relaxation scheme that keeps the coarse-level variables invariant.* Consider the following compatible relaxation iteration (represented here by its corresponding error propagation)

$$\mathbf{e}_{k+1} = (I - S(S^TMS)^{-1}S^TA)\mathbf{e}_k, \tag{5.1}$$

where $S : \mathbb{R}^{n_s} \to \mathbb{R}^n$ is defined, as before, in terms of some restriction operator R. Recall that the coarse-grid variables are defined by $\mathbf{u}_c = R\mathbf{u}$. Since $RS = 0$, we see from (5.1) that $R\mathbf{e}_{k+1} = R\mathbf{e}_k$; that is, the coarse-grid variables are invariant under this iteration. Hence, we need only consider compatible relaxation in the complementary space (in the L-2 sense) via the following iteration:

$$\mathbf{e}_{k+1} = (I - (S^TMS)^{-1}(S^TAS))\mathbf{e}_k. \tag{5.2}$$

Brandt states that *a general measure for the quality of the set of coarse variables is the convergence rate of compatible relaxation.* In the next theorem, we will make this statement rigorous by relating the convergence of the compatible relaxation process in (5.2) to the measure $\mu^\star$ in (3.16) (equivalently, $\mu_\sigma^\star$ in (3.17)).

THEOREM 5.1. *Assume that* $(M+M^T-A)$ *is SPD. Then,*

$$\mu^\star \leq \frac{\Delta^2}{2-\omega}\cdot\frac{1}{1-\rho_s}, \tag{5.3}$$

where constants Δ *and* ω *are as in Lemma 2.3, and where*

$$\rho_s = \left\|(I - M_s^{-1}A_s)\right\|_{A_s}, \tag{5.4}$$

with $M_s = (S^T M S)$ *and* $A_s = (S^T A S)$*. Note that, although we use* ρ *to represent the spectral radius of a matrix, the quantity* ρ_s *is in general only an upper bound for the spectral radius of compatible relaxation; it is equal to the spectral radius when* M *is symmetric.*

Proof. From (3.16), (3.17), and Lemma 2.3, we have that

$$\mu^\star \leq \frac{\Delta^2}{2-\omega}\ \mu^\star_\sigma.$$

But, from (2.11) and Theorem 3.1,

$$\begin{aligned}\mu^\star_\sigma &= \frac{1}{\lambda_{\min}(\sigma(M_s)^{-1}A_s)}\\ &= \max_{\mathbf{v}_s} \frac{\langle M_s\mathbf{v}_s,\ \mathbf{v}_s\rangle}{\langle A_s\mathbf{v}_s,\ \mathbf{v}_s\rangle}\\ &\leq \left\| A_s^{-1/2} M_s A_s^{-1/2} \right\|.\end{aligned}$$

Hence, we have

$$\mu^\star \leq \frac{\Delta^2}{2-\omega} \left\| A_s^{-1/2} M_s A_s^{-1/2} \right\|,$$

and it remains to show that

$$\left\| A_s^{-1/2} M_s A_s^{-1/2} \right\| \leq (1-\rho_s)^{-1}. \tag{5.5}$$

Consider the following symmetric compatible relaxation matrix,

$$H_{ss} = (I - M_s^{-1}A_s)(I - M_s^{-T}A_s).$$

We have that

$$\begin{aligned}\rho(H_{ss}) &= \rho(A_s^{1/2} H_{ss} A_s^{-1/2})\\ &= \rho((I - A_s^{1/2}M_s^{-1}A_s^{1/2})^T(I - A_s^{1/2}M_s^{-1}A_s^{1/2}))\\ &= \left\| (I - A_s^{1/2}M_s^{-1}A_s^{1/2}) \right\|^2\\ &= \left\| (I - M_s^{-1}A_s) \right\|^2_{A_s}\\ &= \rho_s^2.\end{aligned}$$

Noting that H_{ss} can also be written as $I - M_{ss}^{-1}A_s$, where

$$M_{ss}^{-1} = (M_s^{-1} + M_s^{-T} - M_s^{-1}A_s M_s^{-T}),$$

we have that,

$$\rho_s^2 = \rho(H_{ss}) = \max_\lambda |1 - \lambda(M_{ss}^{-1}A_s)| \geq 1 - \lambda_{\min}(M_{ss}^{-1}A_s).$$

Letting $Y_s^{-1} = A_s^{1/2}M_s^{-1}A_s^{1/2}$, one arrives at the coercivity estimate,

$$\begin{aligned}(1-\rho_s^2)\langle \mathbf{v}_s,\ \mathbf{v}_s\rangle &\leq \left\langle M_{ss}^{-1}A_s^{1/2}\mathbf{v}_s,\ A_s^{1/2}\mathbf{v}_s \right\rangle\\ &= \left\langle (Y_s^{-T} + Y_s^{-1} - Y_s^{-1}Y_s^{-T})\mathbf{v}_s,\ \mathbf{v}_s \right\rangle\\ &= 2\left\langle Y_s^{-T}\mathbf{v}_s,\ \mathbf{v}_s\right\rangle - \left\langle Y_s^{-T}\mathbf{v}_s,\ Y_s^{-T}\mathbf{v}_s\right\rangle.\end{aligned} \tag{5.6}$$

Using the Cauchy–Schwarz inequality,

$$\left\langle Y_s^{-T}\mathbf{v}_s,\ \mathbf{v}_s\right\rangle \le \langle \mathbf{v}_s,\ \mathbf{v}_s\rangle^{1/2}\left\langle Y_s^{-T}\mathbf{v}_s,\ Y_s^{-T}\mathbf{v}_s\right\rangle^{1/2},$$

in (5.6), we arrive at

$$\left(\langle \mathbf{v}_s,\ \mathbf{v}_s\rangle^{1/2} - \left\langle Y_s^{-T}\mathbf{v}_s,\ Y_s^{-T}\mathbf{v}_s\right\rangle^{1/2}\right)^2 \le \rho_s^2 \langle \mathbf{v}_s,\ \mathbf{v}_s\rangle .$$

That is,

$$(1-\rho_s)^2 \langle \mathbf{v}_s,\ \mathbf{v}_s\rangle \le \left\langle Y_s^{-T}\mathbf{v}_s,\ Y_s^{-T}\mathbf{v}_s\right\rangle .$$

Adding the left- and right-hand sides of the last estimate and estimate (5.6), one gets,

$$(1-\rho_s) \langle \mathbf{v}_s,\ \mathbf{v}_s\rangle \le \left\langle Y_s^{-T}\mathbf{v}_s,\ \mathbf{v}_s\right\rangle .$$

This implies, letting $\mathbf{v}_s := Y_s\mathbf{v}_s$, that

$$\begin{aligned} \|Y_s\mathbf{v}_s\|^2 &= \langle Y_s\mathbf{v}_s,\ Y_s\mathbf{v}_s\rangle \\ &\le (1-\rho_s)^{-1}\langle \mathbf{v}_s,\ Y_s\mathbf{v}_s\rangle \\ &\le (1-\rho_s)^{-1}\|Y_s\mathbf{v}_s\|\,\|\mathbf{v}_s\| . \end{aligned}$$

Therefore, $\|Y_s\mathbf{v}_s\| \le (1-\rho_s)^{-1}\|\mathbf{v}_s\|$, which implies (5.5), and hence, the result. □

Theorem 5.1 shows that if compatible relaxation is fast to converge (i.e., ρ_s is small), then $\mu^\star$ is small (similarly for $\mu_\sigma^\star$). To use this result in practice as a means of measuring the quality of a given coarse grid, we must be able to efficiently estimate the value of ρ_s in (5.4). One obvious approach for doing this is to run the compatible relaxation iteration in (5.2) and monitor its convergence. In some cases, this may not be feasible. However, in the case where M is derived from a matrix splitting, $A = M - N$, such that M is explicitly available, the iteration in (5.2) is at least computable.

5.1. Compatible Relaxation via Subspace Correction. Another practical form of compatible relaxation is based on the general subspace correction method framework [23], which encompasses both additive and multiplicative Schwarz. Of particular interest is the question of how to define a compatible relaxation variant of overlapping Schwarz. The iteration in (5.2) does not readily admit how to achieve this. In fact, the question of how to define compatible relaxation variants of general subspace correction methods requires some care.

Consider the following additive method

$$I - M^{-1}A; \quad M^{-1} = \sum_i I_i(I_i^T A I_i)^{-1} I_i^T, \tag{5.7}$$

where $I_i : \mathbb{R}^{n_i} \to \mathbb{R}^n$ has full rank, $n_i < n$, and $\mathbb{R}^n = \bigcup_i \mathrm{range}(I_i)$. Define full rank normalized operators S_i and R_i^T such that

$$\mathrm{range}(S_i) = \mathrm{range}(I_i^T S), \tag{5.8}$$

$$\mathrm{range}(R_i^T) = \mathrm{range}(I_i^T R^T). \tag{5.9}$$

In order to define a usable additive version of compatible relaxation, the I_i must be chosen so that the local spaces S_i and R_i^T are orthogonal, i.e., $R_i S_i = 0$. Compatible relaxation is then defined as follows:

$$I - M_{cr}^{-1}A_s; \quad M_{cr}^{-1} = \sum_i S^T I_{s,i}(I_{s,i}^T A I_{s,i})^{-1} I_{s,i}^T S; \quad I_{s,i} = I_i S_i. \tag{5.10}$$

One natural relaxation method that is represented by (5.10) is additive Schwarz. We will discuss this method in more detail below. First, we prove the following lemma and theorem.

LEMMA 5.2. *Assume that we are given the decomposition,*

$$\mathbf{v} = S\mathbf{v}_s + R^T\mathbf{v}_c = \begin{bmatrix} S & R^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_s \\ \mathbf{v}_c \end{bmatrix},$$

such that $RS = 0$ *and* $S^TS = I$*. For any matrix,* M*, we have that*

$$(S^TM^{-1}S)^{-1} = \overline{M}_{Schur} := S^TMS - S^TMR^T(RMR^T)^{-1}RMS.$$

If M *is SPD, then the following also holds:*

$$\left\langle (S^TM^{-1}S)^{-1}\mathbf{v}_s,\ \mathbf{v}_s \right\rangle = \min_{\mathbf{v}_c} \left\langle M(S\mathbf{v}_s + R^T\mathbf{v}_c),\ (S\mathbf{v}_s + R^T\mathbf{v}_c) \right\rangle.$$

Proof. Define the HB matrix

$$\overline{M} := \begin{bmatrix} S & R^T \end{bmatrix}^T M \begin{bmatrix} S & R^T \end{bmatrix} = \begin{bmatrix} \overline{M}_{ss} & \overline{M}_{sc} \\ \overline{M}_{cs} & \overline{M}_{cc} \end{bmatrix}. \tag{5.11}$$

One has, $S^TMS = \overline{M}_{ss}$ by definition. Again, from the definition of $\overline{M}$,

$$\begin{bmatrix} S & R^T \end{bmatrix} \overline{M}^{-1} \begin{bmatrix} S & R^T \end{bmatrix}^T = M^{-1}.$$

Hence,

$$S^TM^{-1}S = S^T \begin{bmatrix} S & R^T \end{bmatrix} \overline{M}^{-1} \begin{bmatrix} S & R^T \end{bmatrix}^T S.$$

Now, using the fact that $RS = 0$ and $S^TS = I$, one gets,

$$S^TM^{-1}S = [\, I \ \ 0 \,]\, \overline{M}^{-1} [\, I \ \ 0 \,]^T.$$

Finally, since

$$\overline{M}^{-1} = \begin{bmatrix} \left(\overline{M}_{\text{Schur}}\right)^{-1} & \star \\ \star & \star \end{bmatrix},$$

one gets,

$$S^TM^{-1}S = [\, I \ \ 0 \,]\, \overline{M}^{-1} [\, I \ \ 0 \,]^T = \left(\overline{M}_{\text{Schur}}\right)^{-1},$$

which implies the first result. The second result follows trivially by noting that

$$\min_{\mathbf{v}_c} \left\langle M(S\mathbf{v}_s + R^T\mathbf{v}_c),\ (S\mathbf{v}_s + R^T\mathbf{v}_c) \right\rangle$$

is a quadratic form in the variable $\mathbf{v}_c$. The minimum is $\left\langle \overline{M}_{\text{Schur}}\mathbf{v}_s,\ \mathbf{v}_s \right\rangle$. ☐

THEOREM 5.3. *Let* M^{-1} *and* M_{cr}^{-1} *be as in* (5.7) *and* (5.10)*, respectively. Define* ω *as in Theorem 5.1, and define*

$$\rho_{cr} = \left\| (I - M_{cr}^{-1}A_s) \right\|_{A_s} = \rho(I - M_{cr}^{-1}A_s). \tag{5.12}$$

Then,

$$\mu^\star \le \frac{1}{2-\omega}\cdot\frac{1}{1-\rho_{cr}}. \tag{5.13}$$

Proof. As before, we can write any vector $\mathbf{e}$ as $\mathbf{e} = S\mathbf{e}_s + R^T\mathbf{e}_c$, for some $\mathbf{e}_s$ and $\mathbf{e}_c$. From (5.8)/(5.9), there exist vectors $\mathbf{e}_{s,i}$ and $\mathbf{e}_{c,i}$ such that $S_i\mathbf{e}_{s,i} = I_i^T S\mathbf{e}_s$ and $R_i^T\mathbf{e}_{c,i} = I_i^T R^T\mathbf{e}_c$. Using this, together with the result in Lemma 5.2 (replacing M^{-1} by $(I_i^T A I_i)$, and R and S by R_i and S_i, respectively), we have

$$\begin{aligned}
\langle M_{cr}^{-1}\mathbf{e}_s,\ \mathbf{e}_s\rangle &= \sum_i \langle (S_i^T I_i^T A I_i S_i)^{-1} S_i^T I_i^T S\mathbf{e}_s,\ S_i^T I_i^T S\mathbf{e}_s\rangle \\
&= \sum_i \langle (S_i^T I_i^T A I_i S_i)^{-1}\mathbf{e}_{s,i},\ \mathbf{e}_{s,i}\rangle \\
&\le \sum_i \left\langle (I_i^T A I_i)^{-1} \left[\ S_i \ \ R_i^T\ \right]\begin{bmatrix}\mathbf{e}_{s,i}\\ \mathbf{e}_{c,i}\end{bmatrix},\ \left[\ S_i \ \ R_i^T\ \right]\begin{bmatrix}\mathbf{e}_{s,i}\\ \mathbf{e}_{c,i}\end{bmatrix}\right\rangle \\
&= \sum_i \left\langle (I_i^T A I_i)^{-1} I_i^T \left[\ S \ \ R^T\ \right]\begin{bmatrix}\mathbf{e}_{s}\\ \mathbf{e}_{c}\end{bmatrix},\ I_i^T\left[\ S \ \ R^T\ \right]\begin{bmatrix}\mathbf{e}_{s}\\ \mathbf{e}_{c}\end{bmatrix}\right\rangle \\
&= \sum_i \langle I_i (I_i^T A I_i)^{-1} I_i^T\mathbf{e},\ \mathbf{e}\rangle \\
&= \langle M^{-1}\mathbf{e},\ \mathbf{e}\rangle.
\end{aligned}$$

Since $\mathbf{e}_c$ was arbitrary, this implies (again, using Lemma 5.2) that

$$\langle M_{cr}^{-1}\mathbf{e}_s,\ \mathbf{e}_s\rangle \le \min_{\mathbf{e}_c}\langle M^{-1}\mathbf{e},\ \mathbf{e}\rangle = \langle (S^T M S)^{-1}\mathbf{e}_s,\ \mathbf{e}_s\rangle = \langle M_s^{-1}\mathbf{e}_s,\ \mathbf{e}_s\rangle.$$

Hence, from (3.16), (3.17), and Lemma 2.3, we have that

$$\begin{aligned}
\mu^\star &\le (2-\omega)^{-1}\ \mu_\sigma^\star \\
&= (2-\omega)^{-1}\ \frac{1}{\lambda_{\min}(M_s^{-1}A_s)} \\
&\le (2-\omega)^{-1}\ \frac{1}{\lambda_{\min}(M_{cr}^{-1}A_s)} \\
&\le (2-\omega)^{-1}\ (1-\rho_{cr})^{-1}.
\end{aligned}$$

□

When the coarse-grid variables are a subset of the fine-grid variables, then we have that $R = [\ 0 \ \ I\ ]$ and $S = [\ I \ \ 0\ ]^T$, and the additive Schwarz method satisfies the criteria for the compatible relaxation in (5.10). To see this, note that, for additive Schwarz, each I_i is a characteristic function over some local subdomain, Ω_i. That is, $I_i\mathbf{w} = \mathbf{w}_i$ on Ω_i and zero outside of Ω_i. From the construction of S_i and R_i^T in (5.8) and (5.9), it is clear that they are also just characteristic functions: R_i^T over the C-pts in Ω_i; and S_i over the F-pts in Ω_i. Hence, $R_i S_i = 0$ for all i.

Multiplicative versions of compatible relaxation are also possible but more difficult to construct, and may not be necessary anyway. Standard Gauss-Seidel and block Gauss-Seidel methods have straightforward compatible relaxation variants, but a general form for multiplicative subspace correction or multiplicative Schwarz (with overlap) is not apparent.

Multiplicative methods are not as practical in the parallel setting, but have better smoothing properties in the sense that ω is usually bounded away from 2 for multiplicative methods without the need for additional smoothing factors. In practice, a good smoother to use is the natural generalization of *F-C* relaxation. That is, (post) smoothing should consist of the above additive compatible relaxation process, followed by the analogous additive compatible relaxation process on the R^T space. Since S^TAS and RAR^T are well-conditioned in some sense, the additive compatible relaxation methods should work well.

5.2. A More General Form of Compatible Relaxation. Although the compatible relaxation methods presented so far provide for many of the traditional relaxation methods, there are still some that may not be represented. In particular, the iteration in (5.2) requires that the matrix M is available *and* that the matrix S^TMS is easily inverted. This may not always be feasible. Additive Schwarz is one such example, albeit one that fortunately has a remedy as described in (5.10). In general, the action of M^{-1} is always available, and motivates us to consider the following compatible relaxation process,

$$\mathbf{e}_{k+1} = (I - (S^TM^{-1}S)(S^TAS))\mathbf{e}_k, \tag{5.14}$$

where, here, S must be normalized so that $S^TS = I_s$, the identity on $\mathbb{R}^{n_s}$. This method is always computable, but must be used with care, as we describe below. First, we state the following result.

THEOREM 5.4. *Assume that the smoother (SPD) M is stable with respect the decomposition $\mathbf{v} = S\mathbf{v}_s + R^T\mathbf{v}_c$, in the sense that for some constant $\gamma \in [0,1)$ the following strengthened Cauchy-Schwarz inequality holds:*

$$\left\langle MS\mathbf{v}_s,\ R^T\mathbf{v}_c\right\rangle \leq \gamma\ \langle MS\mathbf{v}_s,\ S\mathbf{v}_s\rangle^{1/2}\ \left\langle MR^T\mathbf{v}_c,\ R^T\mathbf{v}_c\right\rangle^{1/2},\ \textit{for all}\ \mathbf{v}_s,\mathbf{v}_c. \tag{5.15}$$

Then, the following estimates hold for all $\mathbf{v}_s$,

$$\left\langle (S^TMS)^{-1}\mathbf{v}_s,\ \mathbf{v}_s\right\rangle \leq \left\langle S^TM^{-1}S\mathbf{v}_s,\ \mathbf{v}_s\right\rangle \leq \frac{1}{1-\gamma^2}\ \left\langle (S^TMS)^{-1}\mathbf{v}_s,\ \mathbf{v}_s\right\rangle.$$

In other words the modified compatible relaxation matrix, $(S^TM^{-1}S)$, is spectrally equivalent to the true one, $(S^TMS)^{-1}$.

Proof. Define $\overline{M}$ as in (5.11) in the proof of Lemma 5.2. From the lemma, one trivially has

$$\left\langle (S^TM^{-1}S)^{-1}\mathbf{v}_s,\ \mathbf{v}_s\right\rangle = \left\langle \overline{M}_{\text{Schur}}\mathbf{v}_s,\ \mathbf{v}_s\right\rangle \leq \left\langle S^TMS\mathbf{v}_s,\ \mathbf{v}_s\right\rangle.$$

Replacing M by M^{-1} yields the first inequality. The second inequality follows from the corollary to the strengthened Schwarz inequality,

$$\left\langle \overline{M}_{ss}\mathbf{v}_s,\ \mathbf{v}_s\right\rangle \leq \frac{1}{1-\gamma^2}\ \min_{\mathbf{v}_c}\left\langle \overline{M}\begin{bmatrix}\mathbf{v}_s\\ \mathbf{v}_c\end{bmatrix},\ \begin{bmatrix}\mathbf{v}_s\\ \mathbf{v}_c\end{bmatrix}\right\rangle = \frac{1}{1-\gamma^2}\ \left\langle \overline{M}_{\text{Schur}}\mathbf{v}_s,\ \mathbf{v}_s\right\rangle.$$

Here again, replace M by M^{-1} to get the result. ☐

The above theorem implies the following about the eigenvalues of the corresponding iteration matrix (5.14) and the original compatible relaxation matrix in (5.2):

$$\begin{aligned}\lambda\left(I - (S^TM^{-1}S)A_s\right) &\leq \lambda\left(I - (S^TMS)^{-1}A_s\right)\\ &\leq \gamma^2 + (1-\gamma^2)\lambda\left(I - (S^TM^{-1}S)A_s\right).\end{aligned}$$

Hence, if ρ_g is the spectral radius of $\left(I - (S^T M^{-1} S) A_s\right)$, we arrive at the following result, analogous to the results of Theorem 5.1 and Theorem 5.3:

$$\mu^\star \le \frac{1}{2-\omega} \cdot \frac{1}{1-\gamma^2} \cdot \frac{1}{1-\rho_g}. \tag{5.16}$$

From this, we see that in order to use the compatible relaxation method in (5.14), we must first have an estimate for the size of γ.

In practice, γ can often be estimated locally. This is the case, for example, when M is assembled from small matrices. That is, let $\langle M\mathbf{v},\ \mathbf{v}\rangle = \sum_e \langle M_e \mathbf{v}_e,\ \mathbf{v}_e\rangle = \sum_e \left\langle M_e (I_e)^T \mathbf{v},\ (I_e)^T \mathbf{v}\right\rangle$. Here, $\mathbf{v}|_e = \mathbf{v}_e$. Similarly, for a given $\mathbf{v}_e$ on e, $I_e \mathbf{v}_e$ is the extension of $\mathbf{v}_e$ as zero outside e. Let also $(I_e)^T S = S_e (I_{s,e})^T$, and $(I_e)^T R = R_e (I_{c,\,e})^T$, for S_e, $I_{s,\,e}$, and R_e and $I_{c,\,e}$ supported in e. Then,

$$\left\langle S^T M S \mathbf{v}_s,\ \mathbf{v}_s\right\rangle = \sum_e \left\langle (S_e)^T M_e S_e \mathbf{v}_{s,\,e},\ \mathbf{v}_{s,\,e}\right\rangle.$$

If one can say something about the local matrices $(S_e)^T M_e S_e$ and the local Schur complement $\overline{M}_{e,\,\mathrm{Schur}}$ of $\overline{M}_e = \left[\ S_e \quad R_e^T\ \right]^T M_e \left[\ S_e \quad R_e^T\ \right]$, the maximum of all local γ_e's gives an upper bound for the global γ. This technique is well-known in the two-level HB literature, cf., e.g., R. Bank [2].

A similar approach can be used to estimate γ in the case where M^{-1} is obtained by assembling local matrices. As an example, for additive Schwarz, we have that

$$M^{-1} = \sum_i I_i (I_i^T A I_i)^{-1} I_i^T,$$

where, as described near the end of the previous section, I_i is the characteristic function over some local subdomain, Ω_i. If we have a local estimate of the form

$$\left\langle S_i^T (I_i^T A I_i)^{-1} S_i \mathbf{e}_{s,i},\ \mathbf{e}_{s,i}\right\rangle \le \frac{1}{1-\gamma_i^2} \left\langle (S_i^T I_i^T A I_i S_i)^{-1} \mathbf{e}_{s,i},\ \mathbf{e}_{s,i}\right\rangle,$$

then, using the proof of Theorem 5.13 for the last inequality below, we can show that

$$\begin{aligned}
\left\langle M^{-1} S\mathbf{e}_s,\ S\mathbf{e}_s\right\rangle &= \sum_i \left\langle I_i (I_i^T A I_i)^{-1} I_i^T S\mathbf{e}_s,\ S\mathbf{e}_s\right\rangle \\
&= \sum_i \left\langle S_i^T (I_i^T A I_i)^{-1} S_i \mathbf{e}_{s,i},\ \mathbf{e}_{s,i}\right\rangle \\
&\le \frac{1}{1-\max_i \gamma_i^2} \sum_i \left\langle (S_i^T I_i^T A I_i S_i)^{-1} \mathbf{e}_{s,i},\ \mathbf{e}_{s,i}\right\rangle \\
&\le \frac{1}{1-\max_i \gamma_i^2} \left\langle (S^T M S)^{-1} \mathbf{e}_s,\ \mathbf{e}_s\right\rangle.
\end{aligned}$$

The compatible relaxation method in (5.14) is similar to the habituated compatible relaxation scheme in [17]. The latter has the error propagation

$$\mathbf{e}_{k+1} = (I - S^T M^{-1} A S)\mathbf{e}_k. \tag{5.17}$$

The theoretical result is similar to (5.16). We have the following theorem.

Theorem 5.5. *Assume that the smoother (SPD) M is stable with respect the decomposition $\mathbf{v} = S\mathbf{v}_s + R^T \mathbf{v}_c$, in the sense that for some constant $\gamma \in [0, 1)$ the* *strengthened Cauchy-Schwarz inequality in* (5.15) *holds. Assume that for some constant* $\rho_h < 1$ *the following convergence estimate holds,*

$$\langle A_s \mathbf{e}_{k+1},\ \mathbf{e}_{k+1}\rangle \le \rho_h^2 \langle A_s \mathbf{e}_k,\ \mathbf{e}_k\rangle .$$

Then, the following coercivity estimate holds

$$\delta \langle M_g \mathbf{e}_s,\ \mathbf{e}_s\rangle \le \langle A_s \mathbf{e}_s,\ \mathbf{e}_s\rangle ,$$

where $M_g = (S^T M^{-1} S)^{-1}$ *and* $\delta \ge \frac{1}{2}(1-\rho_h)^2$*. The latter coercivity estimate implies convergence of the compatible relaxation method in* (5.14) *with convergence factor* $\rho_g = 1 - \delta$*).*

Proof. Given $\mathbf{e}_s$, consider the solution $\mathbf{x}$ of the problem,

$$M\mathbf{x} = AS\mathbf{e}_s.$$

The following inequality then follows

$$\begin{aligned} \langle M\mathbf{x},\ \mathbf{x}\rangle &= \left\langle M^{-1/2} AS\mathbf{e}_s,\ M^{1/2}\mathbf{x}\right\rangle \\ &\le \left\langle M^{-1} AS\mathbf{e}_s,\ AS\mathbf{e}_s\right\rangle^{1/2} \langle M\mathbf{x},\ \mathbf{x}\rangle^{1/2} . \end{aligned}$$

This implies that $\langle M\mathbf{x},\ \mathbf{x}\rangle \le \left\langle M^{-1}AS\mathbf{e}_s,\ AS\mathbf{e}_s\right\rangle$, which from Lemma 5.2 and the fact that $2M - A$ is SPD, leads to

$$\begin{aligned} \langle M_g \mathbf{x}_s,\ \mathbf{x}_s\rangle &= \min_{\mathbf{x}_c} \left\langle M(S\mathbf{x}_s + R^T\mathbf{x}_c),\ (S\mathbf{x}_s + R^T\mathbf{x}_c)\right\rangle \\ &\le \langle M\mathbf{x},\ \mathbf{x}\rangle \le \left\langle M^{-1}AS\mathbf{e}_s,\ AS\mathbf{e}_s\right\rangle \le 2\langle A_s\mathbf{e}_s,\ \mathbf{e}_s\rangle . \end{aligned} \tag{5.18}$$

Now, using Cauchy Schwarz and the fact that the habituated compatible relaxation is convergent, one has,

$$\begin{aligned} \langle M_g \mathbf{e}_s,\ \mathbf{e}_s - \mathbf{x}_s\rangle &= \left\langle A_s^{-1/2} M_g \mathbf{e}_s,\ A_s^{1/2}(I - S^T M^{-1} AS)\mathbf{e}_s\right\rangle \\ &\le \rho_h\ \left\langle A_s^{-1} M_g \mathbf{e}_s,\ M_g \mathbf{e}_s\right\rangle^{1/2} \langle A_s \mathbf{e}_s,\ \mathbf{e}_s\rangle^{1/2} \end{aligned}$$

This inequality, using Cauchy-Schwarz and estimate (5.18), implies

$$\begin{aligned} \langle M_g \mathbf{e}_s,\ \mathbf{e}_s\rangle &\le \langle \mathbf{x}_s,\ M_g\mathbf{e}_s\rangle + \rho_h\ \left\langle A_s^{-1} M_g \mathbf{e}_s,\ M_g \mathbf{e}_s\right\rangle^{1/2} \langle A_s \mathbf{e}_s,\ \mathbf{e}_s\rangle^{1/2} \\ &\le \sqrt{2}\ \langle A_s \mathbf{e}_s,\ \mathbf{e}_s\rangle^{1/2} \langle M_g \mathbf{e}_s,\ \mathbf{e}_s\rangle^{1/2} \\ &\quad + \rho_h\ \left\langle A_s^{-1} M_g \mathbf{e}_s,\ M_g \mathbf{e}_s\right\rangle^{1/2} \langle A_s \mathbf{e}_s,\ \mathbf{e}_s\rangle^{1/2} . \end{aligned}$$

Dividing through by $\langle A_s \mathbf{e}_s,\ \mathbf{e}_s\rangle^{1/2} \langle M_g \mathbf{e}_s,\ \mathbf{e}_s\rangle^{1/2}$ one ends up with the inequality,

$$\sqrt{\frac{\langle M_g \mathbf{e}_s,\ \mathbf{e}_s\rangle}{\langle A_s \mathbf{e}_s,\ \mathbf{e}_s\rangle}} \le \sqrt{2}\ + \rho_h \sqrt{\frac{\left\langle A_s^{-1} M_g \mathbf{e}_s,\ M_g \mathbf{e}_s\right\rangle}{\langle M_g \mathbf{e}_s,\ \mathbf{e}_s\rangle}}.$$

Now let,

$$\frac{1}{\delta} = \sup_{\mathbf{e}_s} \frac{\langle M_g \mathbf{e}_s,\ \mathbf{e}_s\rangle}{\langle A_s \mathbf{e}_s,\ \mathbf{e}_s\rangle} = \sup_{\mathbf{e}_s} \frac{\left\langle A_s^{-1} \mathbf{e}_s,\ \mathbf{e}_s\right\rangle}{\left\langle M_g^{-1} \mathbf{e}_s,\ \mathbf{e}_s\right\rangle}.$$

Then, the following inequality is obtained,

$$\frac{1}{\sqrt{\delta}} \leq \sqrt{2} + \rho_h \frac{1}{\sqrt{\delta}}.$$

That is,

$$\frac{1}{\delta} \leq \frac{2}{(1-\rho_h)^2}.$$

□

From the theorem and (5.16), we have the following result for habituated compatible relaxation

$$\mu^\star \leq \frac{1}{2-\omega} \cdot \frac{1}{1-\gamma^2} \cdot \frac{2}{(1-\rho_h)^2}. \tag{5.19}$$

This result is weaker than the previous results for the other compatible relaxation methods. However, as with the method in (5.14), habituated compatible relaxation is always computable. In fact, it is the easiest to implement in practice because it directly involves the global smoother, $I - M^{-1}A$. To see this, note that since S is normalized, the S^T and S in (5.17) can be pulled outside of the parenthesis.

5.3. A Coarsening Algorithm. The above results suggest that compatible relaxation may serve as a useful tool for selecting coarse grids in algebraic multigrid methods. We now present an outline for such a coarsening algorithm in the case where the coarse grid is a subset of the fine grid. That is, consider the case where $R = [\ 0 \ \ I\ ]$ and $S = [\ I \ \ 0\ ]^T$. In the coarsening algorithm, one may apply any of the compatible relaxation methods above, i.e., either (5.2), (5.10), (5.14), or (5.17) to the homogeneous equations

$$(S^T A S)\mathbf{x} = 0, \tag{5.20}$$

with some initial guess, say $\mathbf{x}^0 = (x_i^0)$, where $x_i^0 = 1$ or random positive numbers.

$$
\begin{aligned}
&\text{Initialize } \mathcal{U} = \Omega; \quad \mathcal{C} = \emptyset; \quad \mathcal{F} = \Omega \setminus \mathcal{C} && \text{(5.21a)}\\
&\text{While } \mathcal{U} \neq \emptyset && \text{(5.21b)}\\
&\qquad \text{Do } \nu \text{ compatible relaxation sweeps} && \text{(5.21c)}\\
&\qquad \mathcal{U} = \{i : (|x_i^\nu|/|x_i^{\nu-1}|) > \theta\} && \text{(5.21d)}\\
&\qquad \mathcal{C} = \mathcal{C} \cup \{\text{independent set of } \mathcal{U}\}; \quad \mathcal{F} = \Omega \setminus \mathcal{C} && \text{(5.21e)}
\end{aligned}
$$

This algorithm is similar to what Livne [17] and Brandt [5] use. Note: Instead of adding C-pts, one can also change relaxation.

6. Examples. In this section, we present two examples illustrating the theoretical results of the paper. The first example is a simple anisotropic diffusion problem that demonstrates the ability of the theory (and compatible relaxation) to account for a more general relaxation process; in this case, line relaxation. The example also demonstrates the use of previously developed methods (here, AMGe) for defining adequate interpolation operators in the sense of satisfying **C2** in Theorem 4.2. The second example illustrates how a non-trivial geometric multigrid method for $\mathbf{H}(\text{div})$ fits into the new framework.

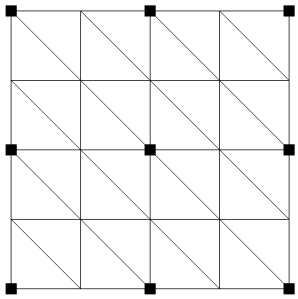

FIG. 6.1. *Uniform grid with triangular elements and standard coarse grid.*

6.1. Compatible Line Relaxation for Anisotropic Diffusion. Consider the grid-aligned anisotropic problem

$$-\epsilon u_{xx} - u_{yy} = f, \ (x, y) \in \Omega = (0, 1)^2,$$

with Dirichlet boundary conditions, discretized on a uniform rectangular grid with mesh-size $h_x = h_y = h = 2^{-\ell}$ as in Figure 6.1. Using piecewise linear elements on triangles, the resulting macro-element matrix for each rectangle is given by,

$$A_e = \epsilon \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

The vertices (nodes) of every rectangle are assumed to have the ordering: (x_i, y_j), (x_{i+1}, y_j), (x_i, y_{j+1}), (x_{i+1}, y_{j+1}); where, $x_i = ih_x$, $y_j = jh_y$, $i, j = 0, 1, \ldots, 2^\ell$.

Consider a block smoother, where the blocks are given by vertical lines of nodes in the grid. That is, consider a line smoother, where the lines are in the "strong" vertical direction. We note that M can be assembled from the same element matrices as A by zeroing some couplings in A_e (namely the ones in the x-direction), yielding

$$M_e = \epsilon \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

Assume standard coarsening, so that $S = [\ I \ \ 0\ ]^T$, where the zero block corresponds to the coarse nodes. We now analyze the convergence rate of the compatible relaxation process in (5.2). Note that S^TMS and S^TAS can also be assembled from local matrices $M_{s,\,e}$ and $A_{s,\,e}$; namely, those obtained from the above matrices in which a row and a column are deleted corresponding to the only coarse node in each rectangular element. Due to symmetry, we delete the last row and last column to get

$$A_{s,\,e} = \epsilon \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \tag{6.1}$$

and

$$M_{s,\,e} = \epsilon \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

It is sufficient to compute the eigenvalues of the generalized eigenvalue problem,

$$A_{s,\,e}\mathbf{x} = \lambda\, M_{s,\,e}\mathbf{x}.$$

This leads to the following cubic equation for λ,

$$\begin{vmatrix} (1-\lambda)(1+\epsilon) & -\epsilon & -(1-\lambda) \\ -\epsilon & (1-\lambda)(1+\epsilon) & 0 \\ -(1-\lambda) & 0 & (1-\lambda)(1+\epsilon) \end{vmatrix} = 0.$$

The roots are,

$$\lambda = 1,\ 1 \pm \sqrt{\frac{\epsilon}{2+\epsilon}}.$$

Hence, the spectrum of the compatible relaxation iteration matrix, $(I - M_s^{-1}A_s)$, is contained in the interval

$$\left[-\sqrt{\frac{\epsilon}{2+\epsilon}},\ \sqrt{\frac{\epsilon}{2+\epsilon}}\right].$$

For $\epsilon \in (0,1]$, this implies that $\rho_s \le 1/\sqrt{3}$. It is well known that linear interpolation is bounded in energy, i.e., it satisfies (4.1) for some constant η independent of ϵ. In fact, for rightangled triangles, one has $\eta = \frac{1}{1-\gamma^2}$ with $\gamma^2 = \frac{1}{2}$, cf. [18]. Hence, from Theorem 4.2 and 2.2, we can conclude that the two-grid method with the above line smoother converges with a rate bounded independent of ϵ (also a well-known fact).

Now, consider the AMGe measure η in (4.5). We know from Corollary 3.3 that

$$\eta \ge \|A\| \frac{1}{\lambda_{\min}(A_{ff})} \tag{6.2}$$

for any interpolation operator, P. Again, because of symmetry, we can bound the minimum eigenvalue of A_{ff} by considering the eigenvalues of the local stiffness matrix with the first and last rows deleted. That is, we can look at the eigenvalues of $A_{s,\,e}$ in (6.1), which satisfy the following cubic equation for λ,

$$\begin{vmatrix} (1+\epsilon-\lambda) & -\epsilon & -1 \\ -\epsilon & (1+\epsilon-\lambda) & 0 \\ -1 & 0 & (1+\epsilon-\lambda) \end{vmatrix} = 0.$$

The roots are

$$\lambda = (1+\epsilon),\ (1+\epsilon) \pm \sqrt{1+\epsilon^2}. \tag{6.3}$$

Hence,

$$\lambda_{\min}(A_{ff}) = (1+\epsilon) - \sqrt{1+\epsilon^2} \le \epsilon, \tag{6.4}$$

which implies that

$$\eta \ge \|A\| \frac{1}{\epsilon} \tag{6.5}$$

for any interpolation operator, P. But, as mentioned earlier in this example, linear interpolation satisfies (4.1) for a constant η independent of ϵ. Hence, although the AMGe measure η in (4.5) also implies (4.1), it is clearly a poor estimate for the latter. Note, however, that we may still use (4.5) to construct good interpolation operators. In particular, the AMGe method can produce linear interpolation for this example; the method is just unable to judge the quality of this interpolation operator when the smoother is line relaxation.

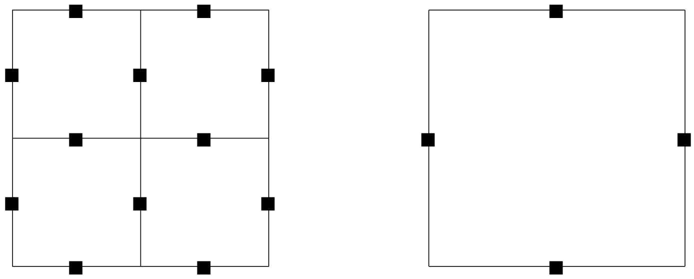

FIG. 6.2. *Coarse rectangle and its refinement. The DOFs of the respective Raviart-Thomas elements are associated with the midpoints of the edges of the elements.*

6.2. Geometric two-grid method for $\mathbf{H}(\text{div})$. The space $\mathbf{H}(\text{div})$ is spanned by vector-functions $\underline{\chi}$ in $(L_2(\Omega))^d$ ($d = 2$ in the present example) whose divergence is also in $L_2(\Omega)$. Consider the Raviart-Thomas finite element discretization [13] of the $\mathbf{H}(\text{div})$ bilinear form,

$$(k^{-1}\underline{\chi},\ \underline{\theta}) + (\nabla \cdot \underline{\chi},\ \underline{\theta}). \tag{6.6}$$

Here, $k = k(x)$ is a given positive coefficient and $(\cdot, \cdot)$ stands for the $L_2(\Omega)$ inner product. The 2D domain Ω is formed from rectangular fine grid elements of mesh size h. The elements are obtained by successive steps of uniform refinement of an initial rectangular coarse mesh. The Raviart-Thomas finite element space of lowest order is spanned locally on every fine-grid rectangle by vector polynomials of the form

$$\begin{bmatrix} ax + b \\ cy + d \end{bmatrix}. \tag{6.7}$$

It is clear that by specifying $\underline{\chi} \cdot \mathbf{n}$ on every edge of the rectangles, then every rectangle has four degrees of freedom, and hence the four coefficients $a,\ b,\ c$, and d are uniquely determined. One also notices that $\underline{\chi} \cdot \mathbf{n}$ on every edge is constant. Hence, $\underline{\chi} \cdot \mathbf{n}$ is continuous across every edge of the fine-grid elements and the vector-function $\underline{\chi}$ is globally in $\mathbf{H}(\text{div})$.

Consider now two triangulations: fine-grid rectangles of mesh size h and coarse-grid rectangles of mesh size $H = 2h$. The degrees of freedom are shown in Fig. 6.2. A standard "Lagrangian" basis of V_h is constructed by choosing, for every fine-grid edge, a function $\underline{\phi}$ which has normal component equal to 1 and zero normal components at the remaining edges. Let T be a coarse rectangle formed by four fine-grid ones. The degrees of freedom (DOFs) of a fine-grid vector $\mathbf{v}$ (w.r.t. the chosen Lagrangian basis) restricted to T can be partitioned into two groups: interior (to T) DOFs and boundary DOFs. The boundary DOFs on every edge of T are given by

$$\begin{bmatrix} \mathbf{v} \cdot \mathbf{n}_1 \\ \mathbf{v} \cdot \mathbf{n}_2 \end{bmatrix} \begin{matrix} \}\ \text{first fine-grid edge} \\ \}\ \text{second fine-grid edge} \end{matrix},$$

and can be decomposed as follows

$$\begin{bmatrix} \mathbf{v} \cdot \mathbf{n}_1 \\ \mathbf{v} \cdot \mathbf{n}_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mathbf{v} \cdot \mathbf{n}_1 - \mathbf{v} \cdot \mathbf{n}_2 \\ \mathbf{v} \cdot \mathbf{n}_2 - \mathbf{v} \cdot \mathbf{n}_1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \mathbf{v} \cdot \mathbf{n}_1 + \mathbf{v} \cdot \mathbf{n}_2 \\ \mathbf{v} \cdot \mathbf{n}_1 + \mathbf{v} \cdot \mathbf{n}_2 \end{bmatrix}.$$

Introduce now the operators acting on vectors spanned by the boundary DOFs:

$$R_B = \frac{1}{\sqrt{2}} [\ I \quad I\ ],\ \text{and}\ S_B = \frac{1}{\sqrt{2}} [\ I \quad -I\ ]^T. \tag{6.8}$$

Next, partition the stiffness matrix A into a 2×2 block form with blocks corresponding to the interior and boundary DOFs. That is,

$$A = \begin{bmatrix} A_{II} & A_{IB} \\ A_{BI} & A_{BB} \end{bmatrix} \begin{matrix} \} \text{ interior fine-grid edges w.r.t. to coarse elements} \\ \} \text{ boundary fine-grid edges w.r.t. to coarse elements} \end{matrix} .$$

Note that A_{II} is block-diagonal with blocks of size 4×4. Denote the reduced matrix (obtained by "static condensation") $A_B = A_{BB} - A_{BI}\,(A_{II})^{-1}\,A_{IB}$. Note that A_B is sparse and explicitly available. For every coarse element edge, fix an ordering of the underlying fine grid edges. This induces a natural partitioning of the boundary DOFs into two groups, corresponding to the above block structure (6.8) of R_B and S_B. Finally, introduce the global decomposition operators,

$$S = \begin{bmatrix} I & -A_{II}^{-1}A_{IB}S_B \\ 0 & S_B \end{bmatrix}, \tag{6.9}$$

and

$$R = [\ 0 \ \ R_B\ ]. \tag{6.10}$$

Clearly, $RS = 0$ and $RR^T = R_B(R_B)^T = I$.

We now choose the following smoother,

$$\begin{aligned} M &= \begin{bmatrix} A_{II} & 0 \\ A_{BI} & \operatorname{diag}(A_B) \end{bmatrix} \begin{bmatrix} I & -A_{II}^{-1}A_{IB} \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} A_{II} & A_{IB} \\ A_{BI} & \operatorname{diag}(A_B) + A_{BI}A_{II}^{-1}A_{IB} \end{bmatrix}. \end{aligned} \tag{6.11}$$

Since M is in a factored form, it is straightforward to implement its inverse action; it involves two actions of the block-diagonal matrix $(A_{II})^{-1}$ and one solve with the scalar diagonal matrix $\operatorname{diag}(A_B)$.

One can see that

$$S^TMS = \begin{bmatrix} A_{II} & 0 \\ 0 & S_B^T \operatorname{diag}(A_B)\ S_B \end{bmatrix}.$$

Similarly,

$$S^TAS = \begin{bmatrix} A_{II} & 0 \\ 0 & S_B^TA_BS_B \end{bmatrix}.$$

The compatible relaxation in (5.2) tells us to look at the matrix

$$\left(S^TMS\right)^{-1}(S^TAS) = \begin{bmatrix} I & 0 \\ 0 & \left(S_B^T \operatorname{diag}(A_B)\ S_B\right)^{-1} S_B^TA_BS_B \end{bmatrix}.$$

Based on a result by Cai, Goldstein and Pasciak [10], one can show that $S_B^TA_BS_B$ is spectrally equivalent to a diagonal matrix. In particular, it is spectrally equivalent to the matrix $\text{const}I_B$, where const is piecewise constant w.r.t. the coarse element edges. This verifies that the respective compatible relaxation gives rise to a well-conditioned matrix $\left(S^TMS\right)^{-1}(S^TAS)$.

It remains to construct a bounded in energy ("approximate harmonic") interpolation operator P. We choose here the P which is naturally defined from the embedding

$V_H \subset V_h$. In operator form, P is the identity. However, in matrix form, its action is computed as follows. Given $\mathbf{v}_H$, consider its four DOFs of the form $\mathbf{v}_H \cdot \mathbf{n}$ for the four edges of every coarse element. These are four constants. Based on these DOFs, one finds the polynomial representation of $\mathbf{v}_H$ on every coarse element. It has the form (6.7). That is, one determines the four constants a, b, c, and d. Then one computes $\mathbf{v}_H \cdot \mathbf{n}$ for all interior fine-grid edges. These, as mentioned above, are also constants (four). Then on every fine grid edge we have specified the fine-grid DOFs $\mathbf{v} \cdot \mathbf{n}$ which are used in the computation.

To prove the energy boundedness of P we proceed as follows. Given $\mathbf{v} \in V_h$. Compute $R\mathbf{v}$. This takes into account only the DOFs which correspond to the boundary (w.r.t. the coarse elements) fine-grid edges. Using function notation it means that we have computed the coarse edge integrals $\int_F \mathbf{v} \cdot \mathbf{n}\ d\varrho$ for every coarse edge F. Based on the four values for every coarse element we construct the unique coarse vector $\mathbf{v}_H = P(R\mathbf{v})$. It has the property that $\int_F \mathbf{v}_H \cdot \mathbf{n}\ d\varrho = \sqrt{2}\ \int_F \mathbf{v} \cdot \mathbf{n}\ d\varrho$. In other words, for any constant function w on a given coarse element T we get

$$\int_{\partial T} w \mathbf{v}_H \cdot \mathbf{n}\ d\varrho = \sqrt{2}\ \int_{\partial T} w \mathbf{v} \cdot \mathbf{n}\ d\varrho.$$

Using the fact that $\nabla w = 0$ on T and the divergence theorem, we get

$$\int_T w \nabla \cdot \mathbf{v}_H\ dx\ dy = \sqrt{2}\ \int_T w \nabla \cdot \mathbf{v}\ dx\ dy.$$

If one introduces the element-wise L_2 projection Q_H onto the space of piecewise constant functions (w.r.t. the coarse elements) , the above identity shows that $\nabla \cdot (PR\mathbf{v}) = \sqrt{2}\ Q_H \nabla \cdot \mathbf{v}$. This immediately implies the inequality,

$$(\nabla \cdot (PR\mathbf{v}),\ \nabla \cdot (PR\mathbf{v})) = 2\ (Q_H \nabla \cdot \mathbf{v},\ Q_H \nabla \cdot \mathbf{v}) \leq 2\ (\nabla \cdot \mathbf{v},\ \nabla \cdot \mathbf{v}).$$

It remains to bound the L_2-norm of $(PR\mathbf{v})$,

$$(PR\mathbf{v},\ PR\mathbf{v}) \leq \eta\ (\mathbf{v},\ \mathbf{v}),$$

for a mesh-independent constant η. We note that

$$\int_T \mathbf{v} \cdot \mathbf{v}\ dxdy \simeq h^2 \sum_{f:\text{ edge of fine-grid element } \tau \subset T} (\mathbf{v} \cdot \mathbf{n}_f)^2.$$

Similarly,

$$\int_T \mathbf{v}_H \cdot \mathbf{v}_H\ dxdy \simeq H^2 \sum_{F:\text{ edge of } T} (\mathbf{v}_H \cdot \mathbf{n}_F)^2.$$

Let $F = f_1 \cup f_2$. Since $\mathbf{v}_H \cdot \mathbf{n}_F = \frac{1}{\sqrt{2}}\ (\mathbf{v} \cdot \mathbf{n}_{f_1} + \mathbf{v} \cdot \mathbf{n}_{f_2})$, hence $(\mathbf{v}_H \cdot \mathbf{n}_F)^2 = \frac{1}{2}\ (\mathbf{v} \cdot \mathbf{n}_{f_1} + \mathbf{v} \cdot \mathbf{n}_{f_2})^2 \leq (\mathbf{v} \cdot \mathbf{n}_{f_1})^2 + (\mathbf{v} \cdot \mathbf{n}_{f_2})^2$. This shows that for a mesh-independent constant η one gets

$$\int_T \mathbf{v}_H \cdot \mathbf{v}_H\ dxdy \leq \eta \int_T \mathbf{v} \cdot \mathbf{v}\ dxdy,$$

which after summation over all coarse elements leads to the required L_2-boundedness of PR. Thus, we get the desired result that PR is bounded in $\mathbf{H}(\text{div})$-norm.

REFERENCES

[1] D. N. Arnold, R. S. Falk, and R. Winther, *Multigrid in* $H(div)$ *and* $H(curl)$, Numerische Mathematik, 85 (2000), pp. 197–217.

[2] R. E. Bank, *Hierarchical bases and the finite element method*, vol. 5 of Acta Numerica, Cambridge University Press, Cambridge, 1996, pp. 1–43.

[3] A. Brandt, *Algebraic multigrid theory: The symmetric case*, in Preliminary Proceedings for the International Multigrid Conference, Copper Mountain, Colorado, April 1983.

[4] A. Brandt, *Algebraic multigrid theory: The symmetric case*, Appl. Math. Comput., 19 (1986), pp. 23–56.

[5] ———, *General highly accurate algebraic coarsening*, Electronic Transactions on Numerical Analysis, 10 (2000), pp. 1–20.

[6] A. Brandt and N. Dinar, *Multigrid solutions to elliptic flow problems*, in Numerical Methods for Partial Differential Equations, S. Parter, ed., Academic Press, New York, 1979, pp. 53–147.

[7] A. Brandt, S. F. McCormick, and J. W. Ruge, *Algebraic multigrid (AMG) for automatic multigrid solutions with application to geodetic computations*. Report, Inst. for Computational Studies, Fort Collins, Colo., October 1982.

[8] ———, *Algebraic multigrid (AMG) for sparse matrix equations*, in Sparsity and Its Applications, D. J. Evans, ed., Cambridge University Press, Cambridge, 1984.

[9] M. Brezina, A. J. Cleary, R. D. Falgout, V. E. Henson, J. E. Jones, T. A. Manteuffel, S. F. McCormick, and J. W. Ruge, *Algebraic multigrid based on element interpolation (AMGe)*, SIAM J. Sci. Comput., 22 (2000), pp. 1570–1592. Also available as LLNL technical report UCRL-JC-131752.

[10] Z. Cai, C. I. Goldstein, and J. E. Pasciak, *Multilevel iteration for mixed finite element systems with penalty*, SIAM J. Sci. Comput., 14 (1993), pp. 1073–1088.

[11] T. Chartier, R. D. Falgout, V. E. Henson, J. E. Jones, T. A. Manteuffel, S. F. McCormick, J. W. Ruge, and P. S. Vassilevski, *Spectral AMGe (ρAMGe)*, SIAM J. Sci. Comput., (2003). To appear. Also available as LLNL technical report UCRL-JC-146369.

[12] A. J. Cleary, R. D. Falgout, V. E. Henson, J. E. Jones, T. A. Manteuffel, S. F. McCormick, G. N. Miranda, and J. W. Ruge, *Robustness and scalability of algebraic multigrid*, SIAM J. Sci. Comput., 21 (2000), pp. 1886–1908. Special issue on the Fifth Copper Mountain Conference on Iterative Methods. Also available as LLNL technical report UCRL-JC-130718.

[13] V. Girault and P. A. Raviart, *Finite element methods for Navier-Stokes equations, theory and algorithms*, Springer-Verlag, New York, 1986.

[14] V. E. Henson and P. S. Vassilevski, *Element-free AMGe: general algorithms for computing interpolation weights in AMG*, SIAM J. Sci. Comput., 23 (2001), pp. 629–650. Also available as LLNL technical report UCRL-JC-139098.

[15] R. Hiptmair, *Multigrid methods for Maxwell's equations*, SIAM J. Numer. Anal., 36 (1998), pp. 204–225.

[16] J. E. Jones and P. S. Vassilevski, *AMGe based on element agglomeration*, SIAM J. Sci. Comput., 23 (2001), pp. 109–133. Also available as LLNL technical report UCRL-JC-135441.

[17] O. E. Livne, *Coarsening by compatible relaxation*, (2003). In preparation.

[18] J. F. Matitre and F. Musy, *The contraction number of a class of two-level method; an exact evaluation for some finite element subspaces and model problems*, in Multigrid Methods, W. Hackbusch and U. Trottenberg, eds., vol. 960 of Lecture Notes in Mathematics, Berlin, 1982, Springer, pp. 534–544. Proc. of the 1st European Multigrid Conference held in Köln-Porz, Germany, 1981.

[19] J. W. Ruge and K. Stüben, *Efficient solution of finite difference and finite element equations by algebraic multigrid (AMG)*, in Multigrid Methods for Integral and Differential Equations, D. J. Paddon and H. Holstein, eds., The Institute of Mathematics and its Applications Conference Series, Clarendon Press, Oxford, 1985, pp. 169–212.

[20] ———, *Algebraic multigrid (AMG)*, in Multigrid Methods, S. F. McCormick, ed., vol. 3 of Frontiers in Applied Mathematics, SIAM, Philadelphia, PA, 1987, pp. 73–130.

[21] U. Trottenberg, C. W. Oosterlee, and A. Schüller, *Multigrid*, Academic Press, London, 2001.

[22] P. Vaněk, J. Mandel, and M. Brezina, *Algebraic multigrid based on smoothed aggregation for second and fourth order problems*, Computing, 56 (1996), pp. 179–196.

[23] J. Xu, *Iterative methods by space decomposition and subspace correction*, SIAM Review, 34 (1992), pp. 581–613.