Balancing Domain Decomposition for Mortar Mixed Finite Element Methods

Ivan Yotov

Gergina Pencheva

Department of Mathematics
University of Pittsburgh

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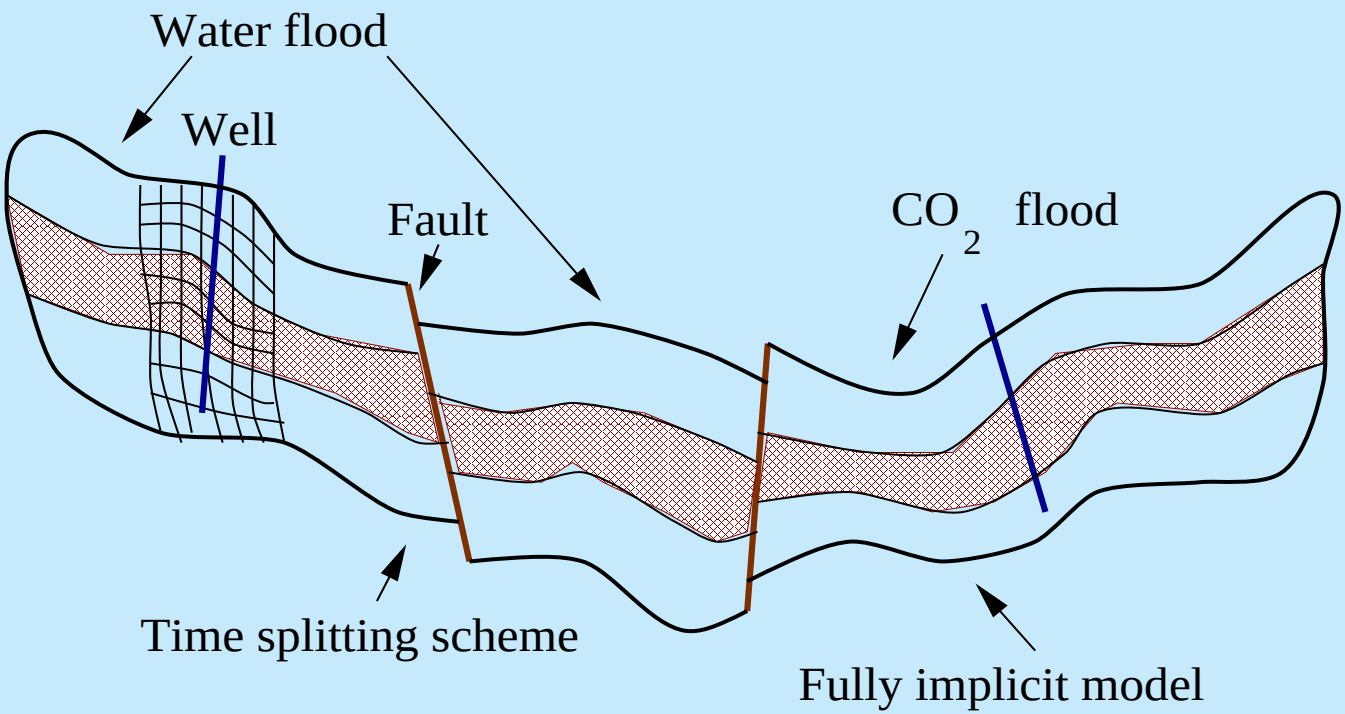
Outline

- Multiblock formulation and discretization for single-phase flow
- Non-overlapping domain decomposition
- Balancing preconditioner for the interface problem
- Condition number estimates
- Numerical experiments
- Extensions to multiphase flow
- Interface Newton-GMRES solver

Multiblock multiphysics approach to flow in porous media



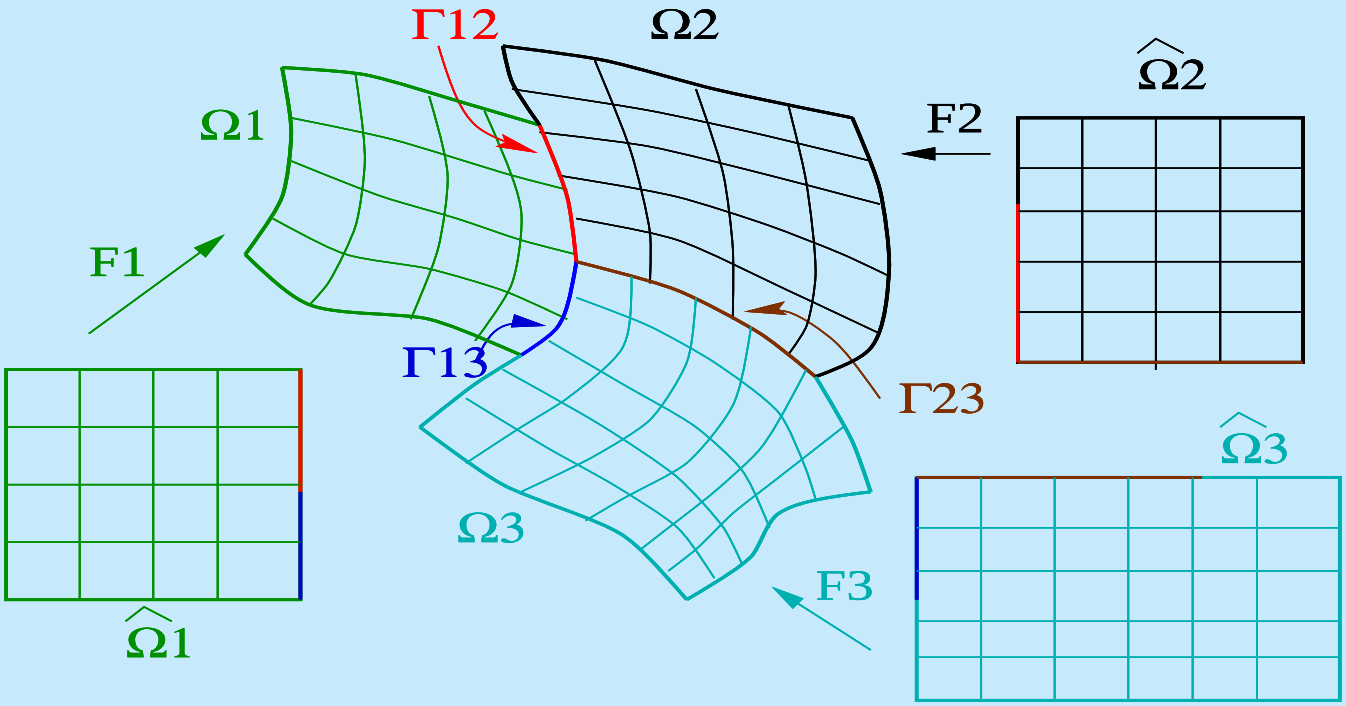
Multiblock gridding strategy

Partition the domain into a series of blocks:

$$\bar{\Omega} = \cup_{i=1}^{n} \bar{\Omega}_i, \quad \Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$$

Each block is covered by a logically rectangular grid.
Consider each block (grid) as the image of a rectangular reference block (grid):

$$F_i : \hat{\Omega}_i \to \Omega_i$$



Features of multiblock subsurface modeling

- Accuracy and local mass conservation
- Irregular geometry - flexible (local) gridding
- Geological features - faults, fractures, layers - non-matching grids
- Multiphysics and local phenomena
- Efficiency - large scale simulations
  - Parallel scalability - domain decompositions
  - Reduced computational cost - different numerical methods on different blocks
  - Local and interface adaptivity

Critical issues:

- Imposing matching conditions across interfaces in a stable and accurate way
- Developing efficient nonlinear interface solvers and preconditioners

Single phase flow model

$\Omega \subset \mathbf{R}^d$ ($d = 2$ or 3) - flow domain

$$\begin{aligned} \mathbf{u} &= -K\nabla p && \text{in } \Omega && \text{(Darcy's law)} \\ \nabla \cdot \mathbf{u} &= q && \text{in } \Omega && \text{(conservation of mass)} \\ \mathbf{u} \cdot \nu &= 0 && \text{on } \partial\Omega && \text{(no flow BC)} \end{aligned}$$

$\mathbf{u}$ - velocity, p - pressure, K - permeability tensor, q - source (wells)

Variational mixed formulation

$$H(\mathrm{div};\Omega) = \{\mathbf{v} : \mathbf{v} \in (L^2(\Omega))^d,\ \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$$

$$\mathbf{V} = \{\mathbf{v} \in H(\mathrm{div};\Omega) : \mathbf{v} \cdot \nu = 0 \text{ on } \partial\Omega\}, \quad W = L^2(\Omega)$$

Find $\mathbf{u} \in \mathbf{V}$, $p \in W$ such that

$$\begin{aligned} (K^{-1}\mathbf{u}, \mathbf{v}) &= (p, \nabla \cdot \mathbf{v}), && \mathbf{v} \in \mathbf{V}, \\ (\nabla \cdot \mathbf{u}, w) &= (q, w), && w \in W. \end{aligned}$$

The mixed finite element method

$\mathcal{T}_h$ - finite element partition
$\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ - mixed finite element spaces

Find $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in W_h$ such that

$$\begin{aligned} (K^{-1}\mathbf{u}_h, \mathbf{v}) &= (p_h, \nabla \cdot \mathbf{v}), & \mathbf{v} &\in \mathbf{V}_h, \\ (\nabla \cdot \mathbf{u}_h, w) &= (q, w), & w &\in W_h. \end{aligned}$$

Properties:

- Simultaneous (accurate) approximation of pressure and velocity
- Local mass conservation: for each element E,

$$w = \begin{cases} 1 \text{ on } E, \\ 0 \text{ otherwise} \end{cases} \Longrightarrow \int_E \nabla \cdot \mathbf{u}_h = \int_E q.$$

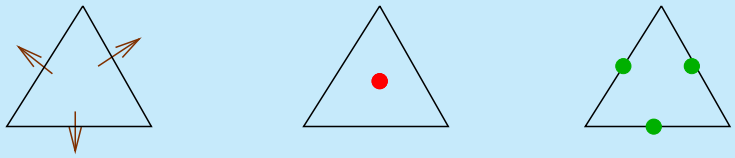
- Continuity of normal flux across element faces: for each $e = \partial E_1 \cap \partial E_2$,

$$\int_e (\mathbf{u}_h|_{E_1} \cdot \nu - \mathbf{u}_h|_{E_2} \cdot \nu) = 0.$$

Raviart-Thomas spaces

- Triangular element:

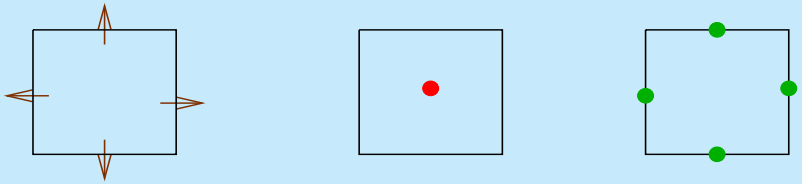
$$\mathbf{V}_h^k(E) = (P_k(E))^2 + \mathbf{x}P_k(E)$$
$$W_h^k(E) = P_k(E)$$



velocity pressure Lagrange multplier

- Rectangular element:

$$\mathbf{V}_h^k(E) = P_{k+1,k}(E) \times P_{k,k+1}(E)$$
$$W_h^k(E) = P_{k,k}(E),$$



velocity pressure Lagrange multplier

Multiblock formulation for single phase flow

$\bar{\Omega} = \cup_{i=1}^{n} \bar{\Omega}_i$
$\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$

On each block Ω_i:

$$\begin{aligned} \mathbf{u} &= -K\nabla p && \text{in } \Omega_i \\ \nabla \cdot \mathbf{u} &= q && \text{in } \Omega_i \\ \mathbf{u} \cdot \nu &= 0 && \text{on } \partial\Omega_i \cap \partial\Omega \end{aligned}$$

On each interface Γ_{ij}:

$$\begin{aligned} p_i &= p_j && \text{on } \Gamma_{ij} \\ [\mathbf{u} \cdot \nu]_{ij} &= 0 && \text{on } \Gamma_{ij} \end{aligned}$$

where

$$\begin{aligned} p_i &= p|_{\partial\Omega_i} \\ [\mathbf{u} \cdot \nu]_{ij} &\equiv \mathbf{u}|_{\Omega_i} \cdot \nu - \mathbf{u}|_{\Omega_j} \cdot \nu \end{aligned}$$

Multiblock (macro-hybrid) discretization

On each block Ω_i:

$\mathcal{T}_{h,i}$ - finite element partition of Ω_i
$\mathcal{T}_{h,i}$ and $\mathcal{T}_{h,j}$ - possibly non-matching on Γ_{ij}
$(\mathbf{u}_{h,i}, p_{h,i}) \in \mathbf{V}_{h,i} \times W_{h,i}$ - mixed finite element approximation on $\mathcal{T}_{h,i}$

On each interface Γ_{ij}:

Interface finite element grid $\mathcal{T}_{h,ij}$ - possibly different from the grids on Ω_i and Ω_j.
Mortar finite element space $M_{h,ij}$ on $\mathcal{T}_{h,ij}$.

Interface pressure is approximated by λ_h in the mortar space $M_{h,ij}$.

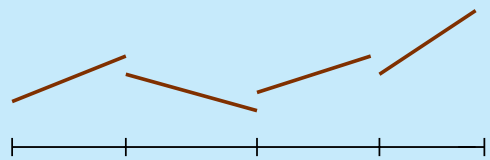
Flux continuity is imposed weakly:

$$\int_{\Gamma_{ij}} [\mathbf{u} \cdot \nu]_{ij} \mu \, d\sigma = 0, \quad \forall \mu \in M_{h,ij}.$$

Mortar finite element spaces for $\mathbf{RT}_0$

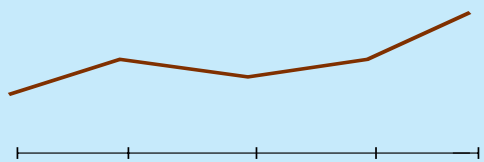
Matching interface Γ_{ij}: Use the standard piecewise constant Lagrange multiplier space.
Non-matching interface Γ_{ij}: The piecewise constant space leads to only sub-optimally convergent method.
Instead, use piecewise linears:

$$M^d_{h,ij} = \{\mu \in L^2(\Gamma_{ij}) : \mu|_e \text{ is (bi)linear}\}$$



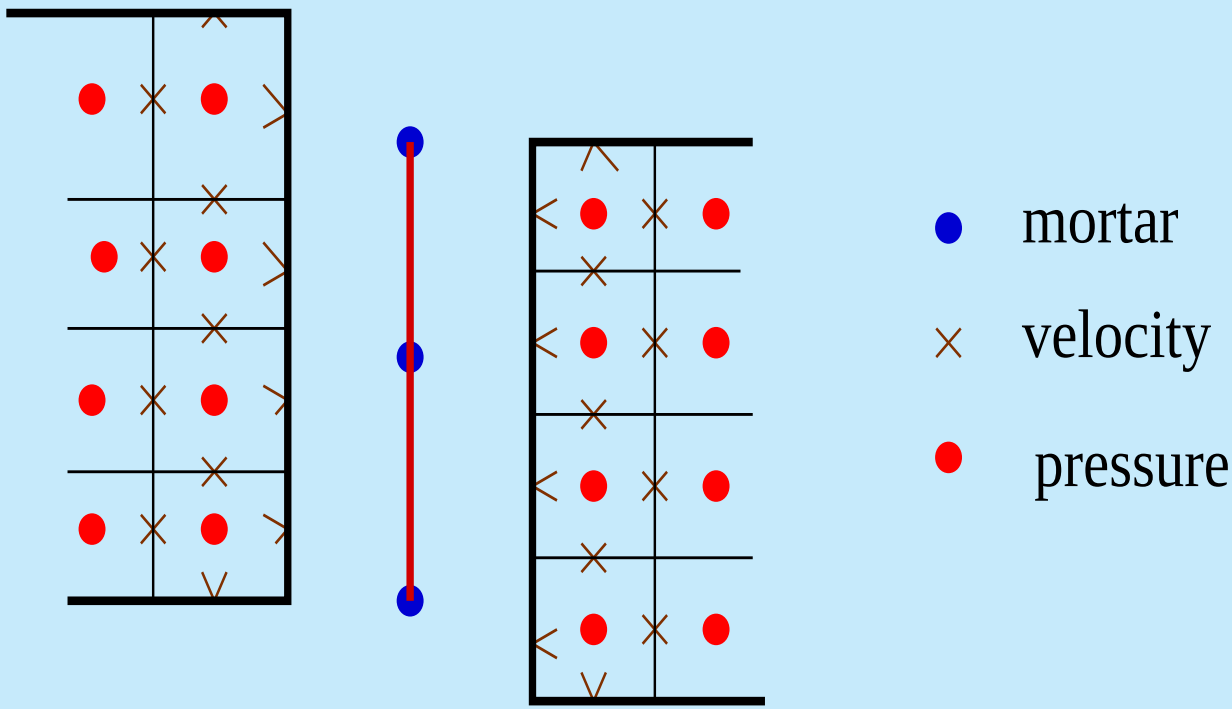
and

$$M^c_{h,ij} = M^d_{h,ij} \cap C^0(\Gamma_{ij}).$$



$M_{h,ij} = M^d_{h,ij}$ or $M^c_{h,ij}$ - a mortar finite element space on Γ_{ij}.

Multiblock discretization spaces



$$\mathbf{V}_h = \bigoplus_{i=1}^{n} \mathbf{V}_{h,i}, \quad W_h = \bigoplus_{i=1}^{n} W_{h,i}, \quad M_h = \bigoplus_{0 \le i < j \le n} M_{h,ij}$$

The mortar mixed finite element method

Find $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in W_h$, $\lambda_h \in M_h$ s.t. for $1 \le i \le n$

$$
\begin{aligned}
&(K^{-1}\mathbf{u}_h, \mathbf{v})_{\Omega_i} = (p_h, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda_h, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i}, && \mathbf{v} \in \mathbf{V}_{h,i},\\
&(\nabla \cdot \mathbf{u}_h, w)_{\Omega_i} = (q, w)_{\Omega_i}, && w \in W_{h,i},\\
&\sum_{i=1}^{n} \langle \mathbf{u}_h \cdot \nu_i, \mu \rangle_{\Gamma_i} = 0, && \mu \in M_h.
\end{aligned}
$$

Stability condition on $M_{h,ij}$:

$$\mathcal{Q}_{h,i}\mu = \mathcal{Q}_{h,j}\mu = 0 \Rightarrow \mu = 0, \quad \forall \mu \in M_{h,ij},$$

where $\mathcal{Q}_{h,i} : M_{h,ij} \to \mathbf{V}_{h,i} \cdot \nu|_{\Gamma_{ij}}$ is the L^2-projection:

$$\langle \mu - \mathcal{Q}_{h,i}\mu, \mathbf{v} \cdot \nu \rangle = 0, \quad \mathbf{v} \in \mathbf{V}_{h,i}.$$

Lemma: There exists a unique solution $(\mathbf{u}_h, p_h, \lambda_h)$ such that

$$\|u_h\|_{\mathbf{V}} + \|p_h\|_W + \|\lambda_h\|_{L^2(\Gamma)} \le C\|q\|_{L^2(\Omega)}.$$

Convergence results

Theorem (Arbogast, Wheeler, Cowsar, Y.)
Assume that the grids are quasi-uniform and for each $\mu \in \Lambda_h$

$$\|\mu\|_{0,\Gamma_{i,j}} \le C(\|\mathcal{Q}_{h,i}\mu\|_{0,\Gamma_{i,j}} + \|\mathcal{Q}_{h,j}\mu\|_{0,\Gamma_{i,j}}).$$

Then

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le C \sum_{i=1}^{n} (\|p\|_{r+1,\Omega_i} + \|\mathbf{u}\|_{r,\Omega_i}) h^r, \quad 0 < r \le k+1,$$

$$\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_0 \le C \sum_{i=1}^{n} \|\nabla \cdot \mathbf{u}\|_{r,\Omega_i} h^r, \quad 0 < r \le l+1,$$

$$|||\mathbf{u} - \mathbf{u}_h||| \le C \sum_{i=1}^{n} (\|p\|_{r+3/2,\Omega_i} + \|\mathbf{u}\|_{r+1/2,\Omega_i}) h^{r+1/2}, \quad 0 < r \le k+1,$$

$$\|\hat{p} - p_h\|_0 \le C \sum_{i=1}^{n} (\|p\|_{r+1,\Omega_i} + \|\mathbf{u}\|_{r,\Omega_i} + \|\nabla \cdot \mathbf{u}\|_{r,\Omega_i}) h^{r+1},$$

$$\|p - p_h\|_0 \le C \sum_{i=1}^{n} (\|p\|_{r+1,\Omega_i} + \|\mathbf{u}\|_{r,\Omega_i} + \|\nabla \cdot \mathbf{u}\|_{r,\Omega_i}) h^{r},$$

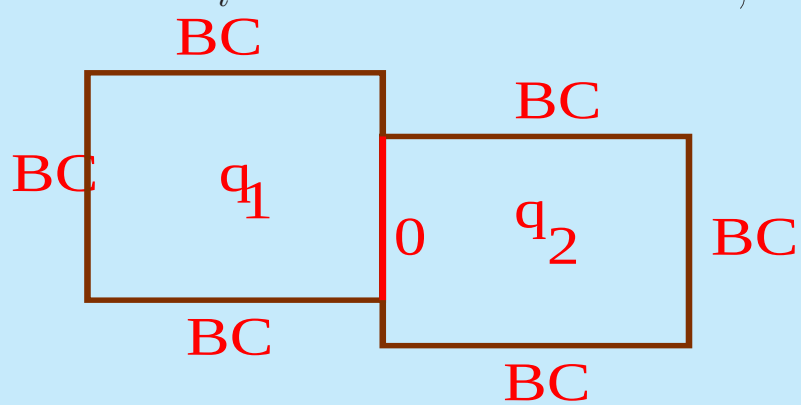
$$0 < r \le \min(k+1, l+1).$$

Here $|||\cdot|||$ denote the discrete L^2-norm along the Gaussian lines.

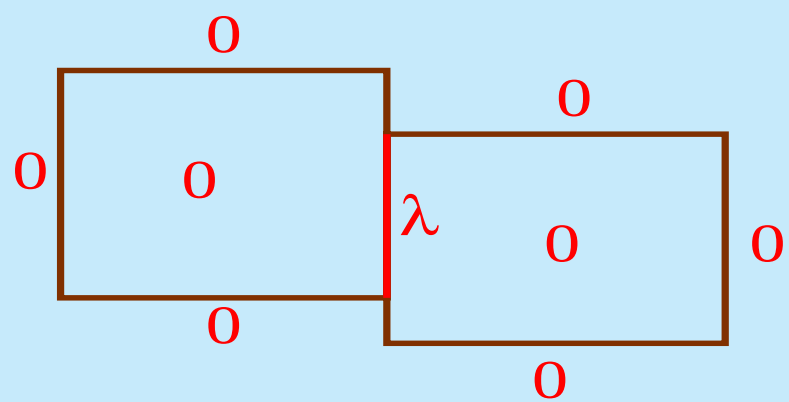
Parallel domain decomposition

[Glowinski, Wheeler; Y] Two types of subdomain problems:

$(\bar{\mathbf{u}}_h|_{\Omega_i}, \bar{p}_h|_{\Omega_i})$ is the solution on Ω_i with zero mortar data, correct source and BC's.



$(\mathbf{u}_h^*(\lambda)|_{\Omega_i}, p_h^*(\lambda)|_{\Omega_i})$ is the solution on Ω_i with mortar data $\lambda \in M_h$, zero source and BC's.



Reduction to an interface problem

Define $s_h : L^2(\Gamma) \times L^2(\Gamma) \to \mathbf{R}$ by

$$s_h(\lambda, \mu) = -\sum_{i,j} \int_{\Gamma_{i,j}} [\mathbf{u}_h^*(\lambda) \cdot \nu]_{ij} \, \mu \, d\sigma$$

Define $g_h : L^2(\Gamma) \to \mathbf{R}$ by

$$g_h(\mu) = \sum_{i,j} \int_{\Gamma_{i,j}} [\bar{\mathbf{u}}_h \cdot \nu]_{ij} \, \mu \, d\sigma$$

The solution $(\mathbf{u}_h, p_h, \lambda_h)$ to the original problem satisfies

$$s_h(\lambda_h, \mu) = g_h(\mu), \qquad \mu \in M_h,$$

with

$$\mathbf{u}_h = \mathbf{u}_h^*(\lambda_h) + \bar{\mathbf{u}}_h, \quad p_h = p_h^*(\lambda_h) + \bar{p}_h.$$

Lemma

$s_h(\cdot, \cdot)$ is symmetric and positive semi-definite on M_h.

Solve the interface problem

$$s_h(\lambda_h, \mu) = g_h(\mu), \qquad \mu \in M_h$$

in parallel by using conjugate gradient or multigrid.

Iterative interface solver

Evaluation of the bilinear form on each iteration:

- Project λ onto $\mathbf{V}_{h,i} \cdot \nu$: $\lambda \to \mathcal{Q}_{h,i}\lambda$
- Solve in parallel subdomain problems with Dirichlet data $\mathcal{Q}_{h,i}\lambda$ on the interior interfaces to compute the fluxes $\mathbf{u}_h^*(\lambda) \cdot \nu$
- Project the fluxes back to the mortar space M_h to compute the jumps across the interfaces

Algebraic system

$$\begin{pmatrix} A & B & L \\ B^T & 0 & 0 \\ L^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ q \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} R & L \\ L^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} \tilde{q} \\ 0 \end{pmatrix}$$

$\mathbf{x}$ - subdomain unknowns; λ - interface unknowns

Matrix is symmetric but indefinite. Form the Shur complement system:

$$L^T R^{-1} L \lambda = L^T R^{-1} \tilde{q} \tag{1}$$

Interface problem

Define, for $i = 1, ..., n$, $S_i : M_h \to M_h$ and $s_i : L^2(\Gamma) \times L^2(\Gamma) \to \mathbf{R}$ such that for all $\lambda, \mu \in M_h$

$$\langle S_i \lambda, \mu \rangle = s_i(\lambda, \mu) = -\langle \mathbf{u}_i^*(\lambda) \cdot \nu_i, \mu \rangle_{\Gamma_i}.$$

S_i is a Dirichlet-to-Neumann map:

$$S_i \lambda = -\mathcal{Q}_i^T (\mathbf{u}_i^*(\lambda) \cdot \nu_i)$$

Define $S : M_h \to M_h$ and $s : L^2(\Gamma) \times L^2(\Gamma) \to \mathbf{R}$ such that

$$S = \sum_{i=1}^{n} S_i, \quad s(\lambda, \mu) = \sum_{i=1}^{n} s_i(\lambda, \mu).$$

The interface problem is

$$S\lambda = g \quad \text{or} \quad s(\lambda, \mu) = g(\mu), \quad \forall \mu \in M_h.$$

Balancing preconditioner

Balancing preconditioner background

Interface formulation: Glowinski and Wheeler (1988)

Neumann-Neumann preconditioner:

Bourgat, Glowinski, Le Tallec, and Vidrascu (1989)
De Roeck and Le Tallec (1991)
Le Tallec (1993)
Achdou, Le Tallec, Nataf, and Vidrascu (2000)

Balancing preconditioner:

Mandel (1993)
Mandel and Brezina (1996)
Cowsar, Mandel, and Wheeler (1995)

Interface Neumann-Neumann preconditioner

Define a partition of unity D_i: $supp(D_i) \subset \Gamma_i$ and $\Sigma_{i=1}^n D_i\lambda = \lambda$, $\forall \lambda \in M_h$.

ALGORITHM:

Given $r \in M_h$, define $M_{NN}^{-1}r$ as follows:

1. $r_i = D_i^T r$ (distribute r to subdomains).
2. Solve $S_i\lambda_i = r_i$ for $\lambda_i \in M_h$.
3. $M_{NN}^{-1}r = \Sigma_{i=1}^n D_i\lambda_i$ (average local solutions).

$$M_{NN}^{-1} = \sum_{i=1}^n D_i S_i^{-1} D_i^T$$

$S_i^{-1} : r_i \to \lambda_i$ is a Neumann-to-Dirichlet map and requires a local Neumann solve:
Find $\mathbf{u}_i \in V_{h,i}^0$, $p_i \in W_{h,i}$, $\lambda_i \in M_{h,i}$ such that

$$\begin{aligned}
&(K^{-1}\mathbf{u}_i, \mathbf{v})_{\Omega_i} = (p_i, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda_i, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i}, && \mathbf{v} \in \mathbf{V}_{h,i}^0,\\
&(\nabla \cdot \mathbf{u}_i, w)_{\Omega_i} = 0, && w \in W_{h,i},\\
&\langle \mathbf{u}_i \cdot \nu_i, \mu \rangle_{\Gamma_i} = \langle r_i, \mu \rangle_{\Gamma_i}, && \mu \in M_{h,i}.
\end{aligned}$$

Problems: S_i may be singular; No global exchange of information.

Balancing preconditioner

1. Balance residuals so local problems $S_i\lambda_i = r_i$ are solvable (modulo Null S_i).
2. Result should not depend on the specific shoice of local solutions.
3. Parallel scalability - global transfer of information through a coarse solve.

$S_i\lambda_i = r_i$ is solvable if $r_i \perp \text{Null} S_i$.

$$\text{Null} S_i = \begin{cases} \{\text{const}\} & \text{if full Neumann} \\ \emptyset & \text{otherwise} \end{cases}$$

Define Z_i: Null $S_i \subseteq Z_i$ (Take $Z_i = \{\text{const}\}$).
Coarse space:

$$M_H = \{\lambda \in M_h : \lambda = \sum_{i=1}^{n} D_i\zeta_i,\ \zeta_i \in Z_i\}.$$

dim $M_H \leq n$ (number of subdomains)
A residual r is balanced (local problems are solvable) if

$$\langle r, \mu_H\rangle_\Gamma = 0, \quad \forall \mu_H \in M_H.$$

Balancing r: $r^{bal} = r - Sr_H$ where $r_H \in M_H$ is found by solving a coarse problem

$$s(r_H, \mu_H) = \langle r, \mu_H\rangle_\Gamma, \quad \forall \mu_H \in M_H.$$

Balancing preconditioner (Cont.)

ALGORITHM:

Given $r \in M_h$, define $M_{bal}^{-1} r$ as follows:

1. Solve a coarse problem

$$s(r_H, \mu_H) = \langle r, \mu_H \rangle_\Gamma, \quad \forall \mu_H \in M_H,$$

and balance residual

$$r^{bal} = r - S r_H.$$

2. $r_i = D_i^T r^{bal}$ (distribute r^{bal} to subdomains).
3. Solve local problems for $\lambda_i \in M_{h,i}$:

$$S_i \lambda_i = r_i.$$

4. $\lambda = \Sigma_{i=1}^n D_i \lambda_i$ (average local solutions).
5. Solve a coarse problem

$$s(\lambda_H, \mu_H) = \langle r, \mu_H \rangle_\Gamma - s(\lambda, \mu_H), \quad \forall \mu_H \in M_H,$$

and balance local solutions

$$M_{bal}^{-1} r = \lambda + \lambda_H.$$

Condition number estimate

Assume that

$$c\alpha_i \xi^T \xi \le \xi^T K(x) \xi \le C\alpha_i \xi^T \xi \quad \forall \xi \in \mathbf{R}^d, \forall x \in \Omega_i.$$

Lemma

$$\frac{1}{\alpha_j} s_j(\lambda_i, \lambda_i) \le \frac{C}{\alpha_j}(1 + \log(H/h))^2 s_i(\lambda_i, \lambda_i).$$

Proof uses:

$$c|\mathcal{I}^{\partial\Omega_i} \mathcal{Q}_{h,i}\mu|^2_{1/2,\partial\Omega_i} \le \frac{1}{\alpha_i} s_i(\mu, \mu) \le C|\mathcal{I}^{\partial\Omega_i} \mathcal{Q}_{h,i}\mu|^2_{1/2,\partial\Omega_i}, \quad \mu \in M_h$$

and

$$\|\mathcal{I}^{\partial\Omega_j} \mathcal{Q}_{h,j}\mu\|_{1/2,\Gamma_{i,j}} \simeq \|\mathcal{I}^{\partial\Omega_i} \mathcal{Q}_{h,i}\mu\|_{1/2,\Gamma_{i,j}}, \quad \mu \in M_h$$

Choose weights on Ω_i

$$(D_i \lambda_i)(x) = \frac{\alpha_i}{\alpha_i + \alpha_j} \lambda_i(x), \quad x \in \partial\Omega_i \cap \partial\Omega_j.$$

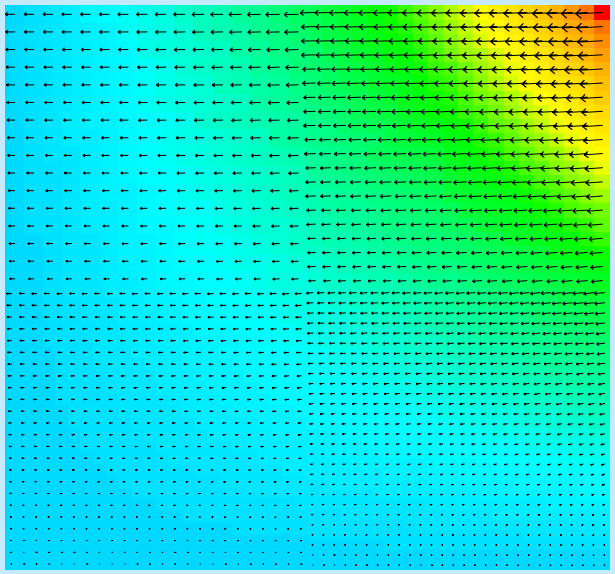
Theorem

$$\mathrm{cond}(M_{bal}^{-1} S) \le C(1 + \log(H/h))^2,$$

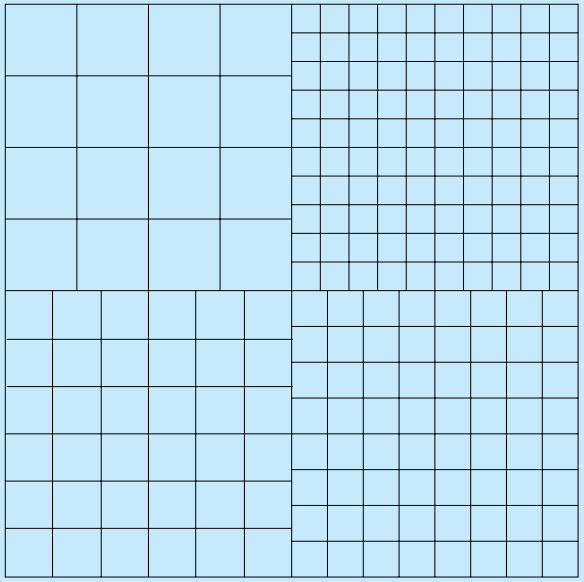
where C does not depend on h, H, and jumps in K.

Computational Test 1

$$p(x,y,z) = x^3y^2 + \sin(xy), \quad K = \begin{pmatrix} 10 + 5\cos(xy) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



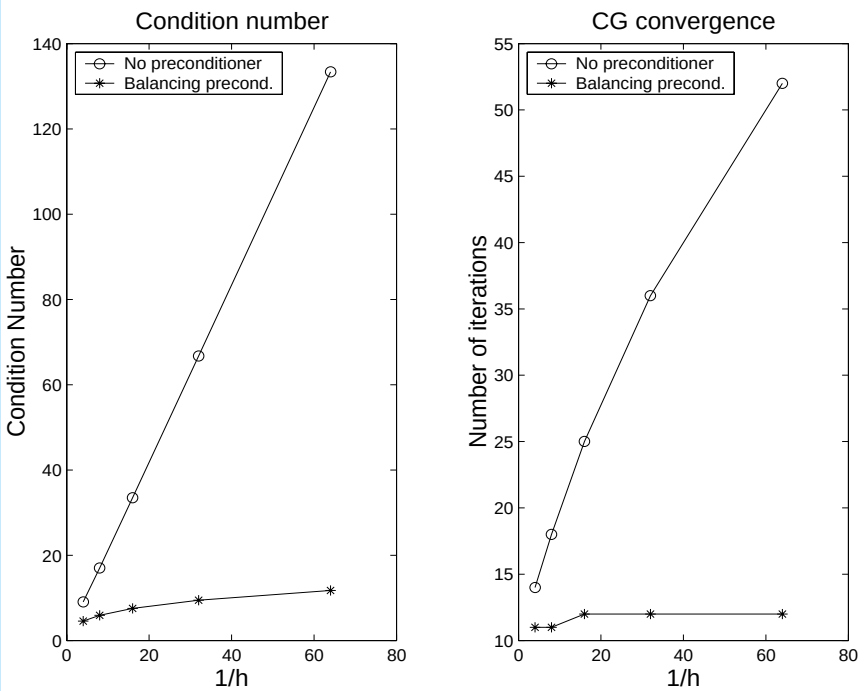
Computed pressure and velocity

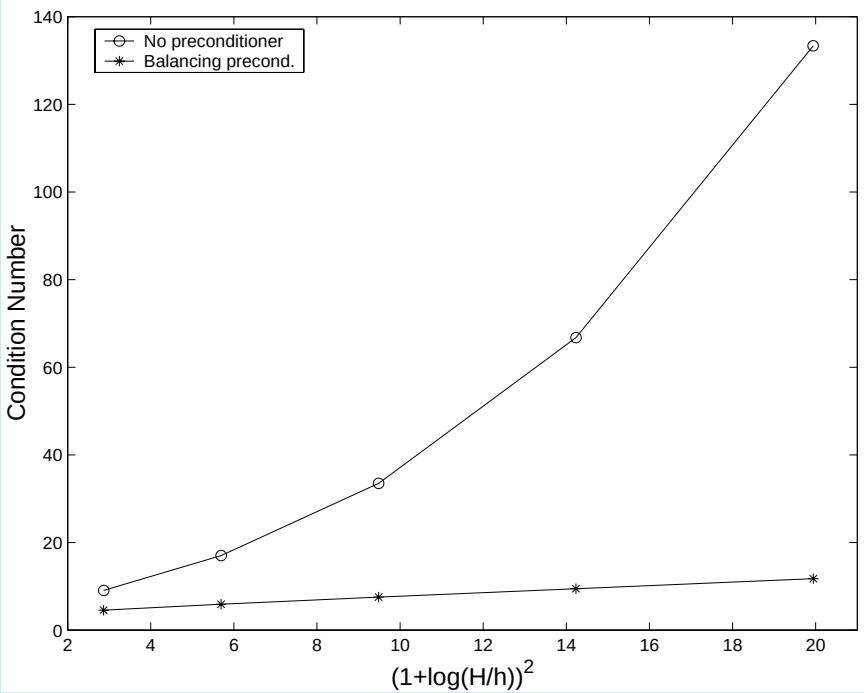


Initial grids

Convergence for Test 1

$1/h$	$BalCG$		CG	
	cond.	*iter.*	*cond.*	*iter.*
4	4.54520	11	9.04823	14
8	5.90177	11	17.0075	18
16	7.54221	12	33.5087	25
32	9.44828	12	66.7478	36
64	11.7220	12	133.354	52



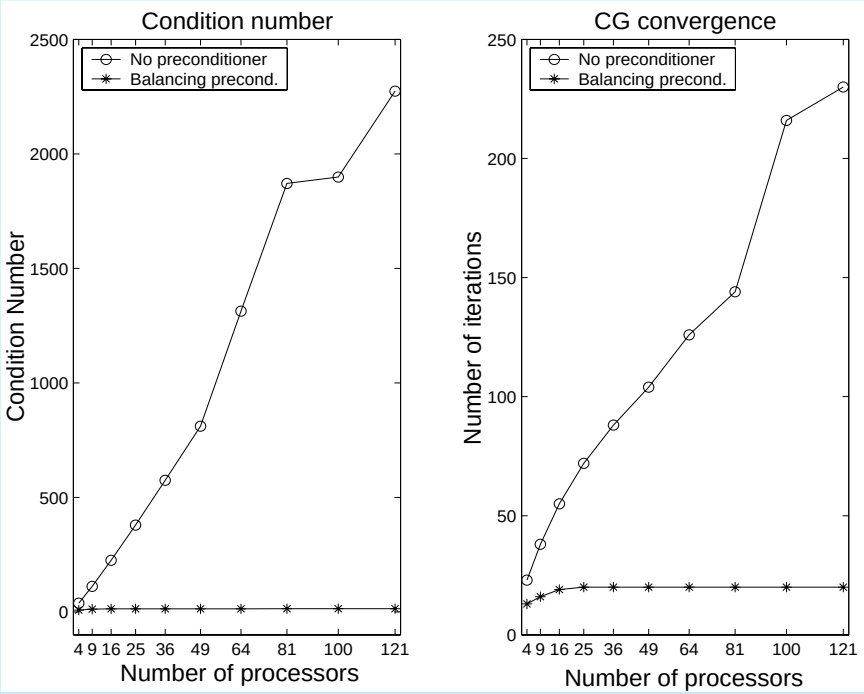


Computational Test 2

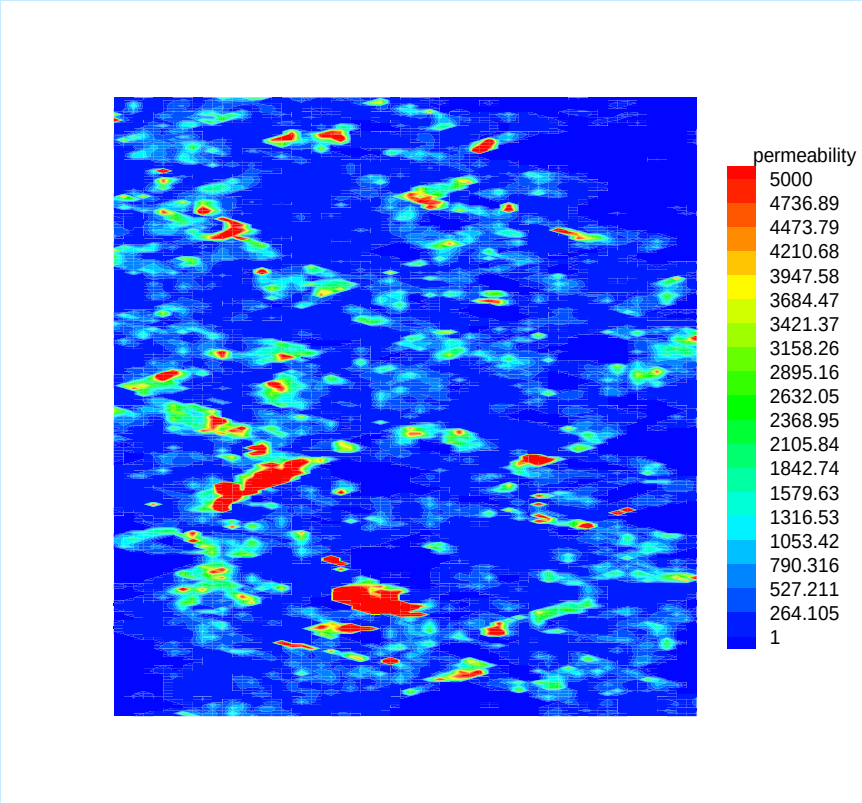
$$p(x,y,z) = x^3y^4 + x^2 + \sin(xy)\cos(y), \quad K = \begin{pmatrix} (x+1)^2 + y^2 & 0 & 0 \\ 0 & (x+1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2 x 2 to 11 x 11 subdomains with grids 14 x 18, 12 x 20 in checkboard fashion

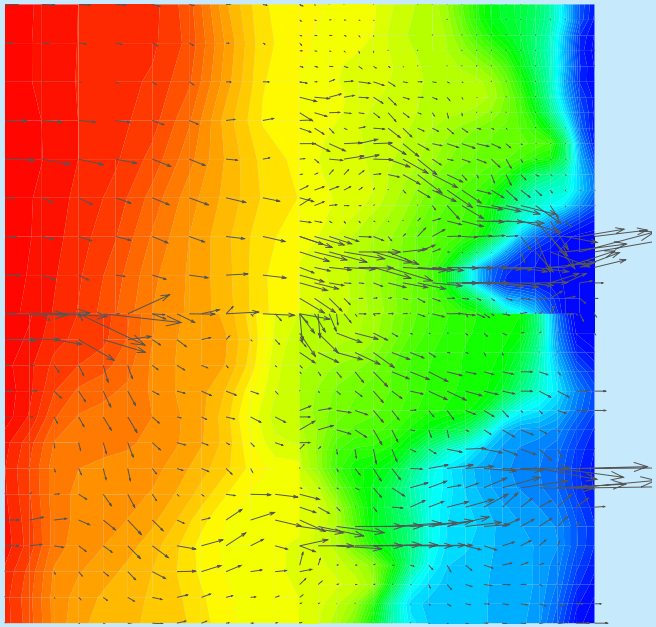
$1/h$	*BalCG*		*CG*	
	cond.	*iter.*	*cond.*	*iter.*
4	8.16598	13	38.2574	23
9	12.2771	16	111.967	38
16	13.2133	19	225.357	55
25	13.4436	20	379.496	72
36	13.6084	20	574.747	88
49	13.6382	20	811.262	104
64	13.7507	20	1313.49	126
81	13.8927	20	1871.29	144
100	13.9882	20	1898.91	216
121	14.0497	20	2274.17	230



Computational Test 3



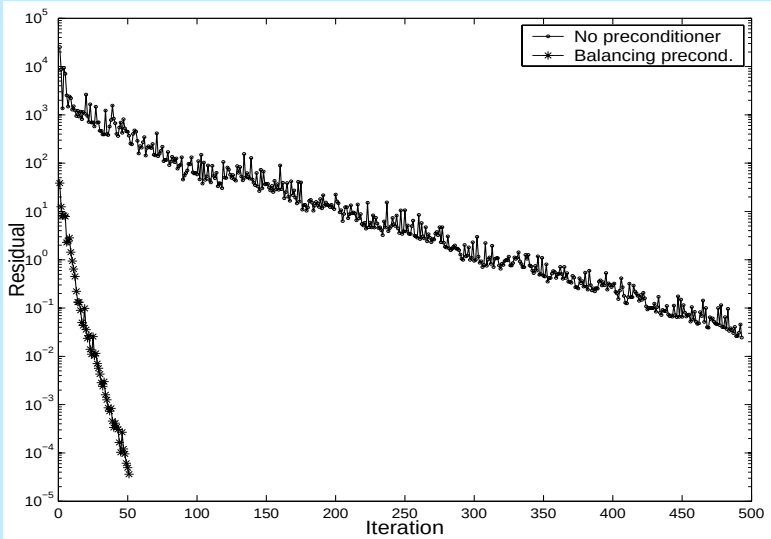
Permeability Tensor K

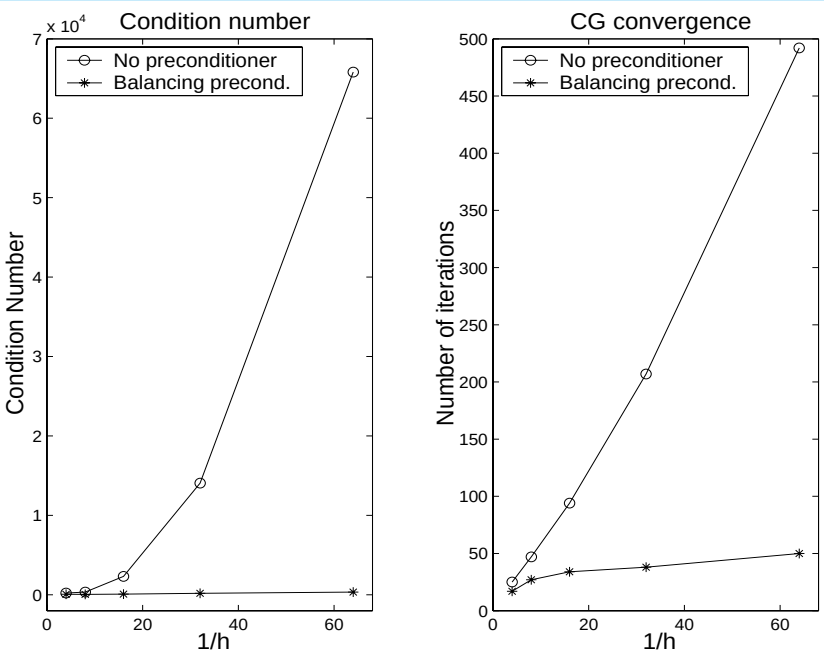


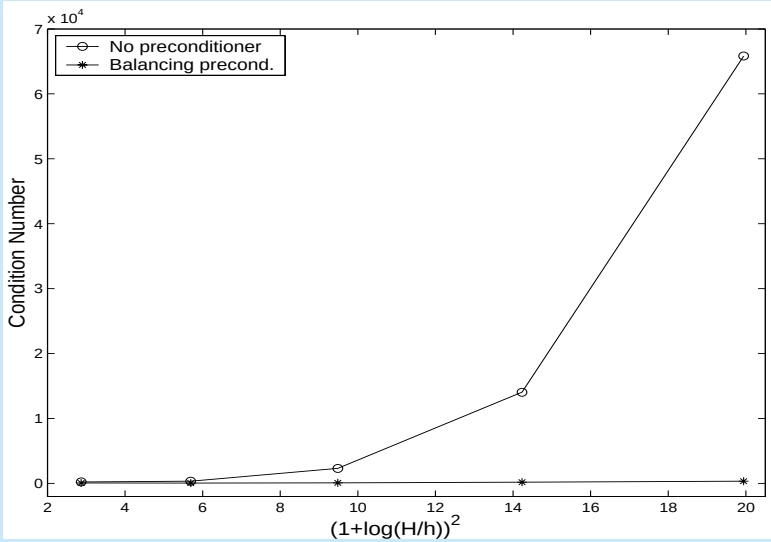
Approximate solution

Convergence for Test 3

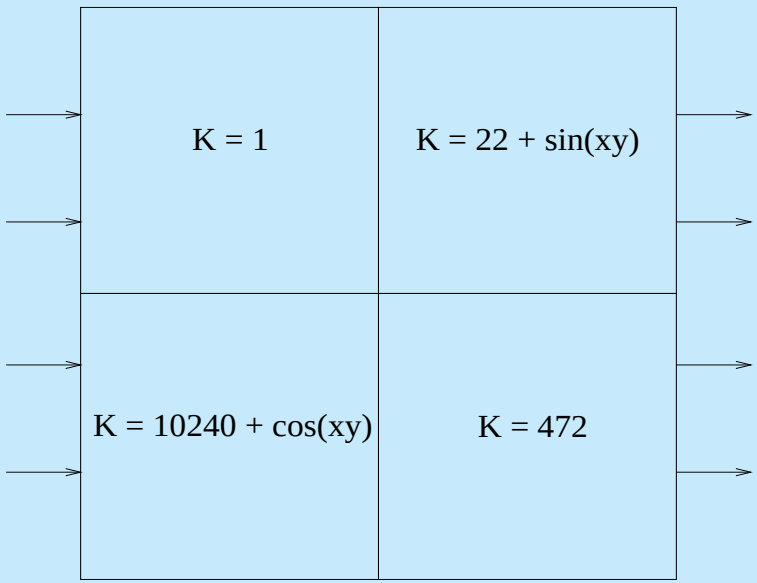
$1/h$	*BalCG*		*CG*	
	cond.	*iter.*	*cond.*	*iter.*
4	63.4544	17	222.195	25
8	53.8312	27	338.236	47
16	90.2661	34	2313.30	94
32	192.754	38	14049.8	207
64	336.072	50	65824.2	492

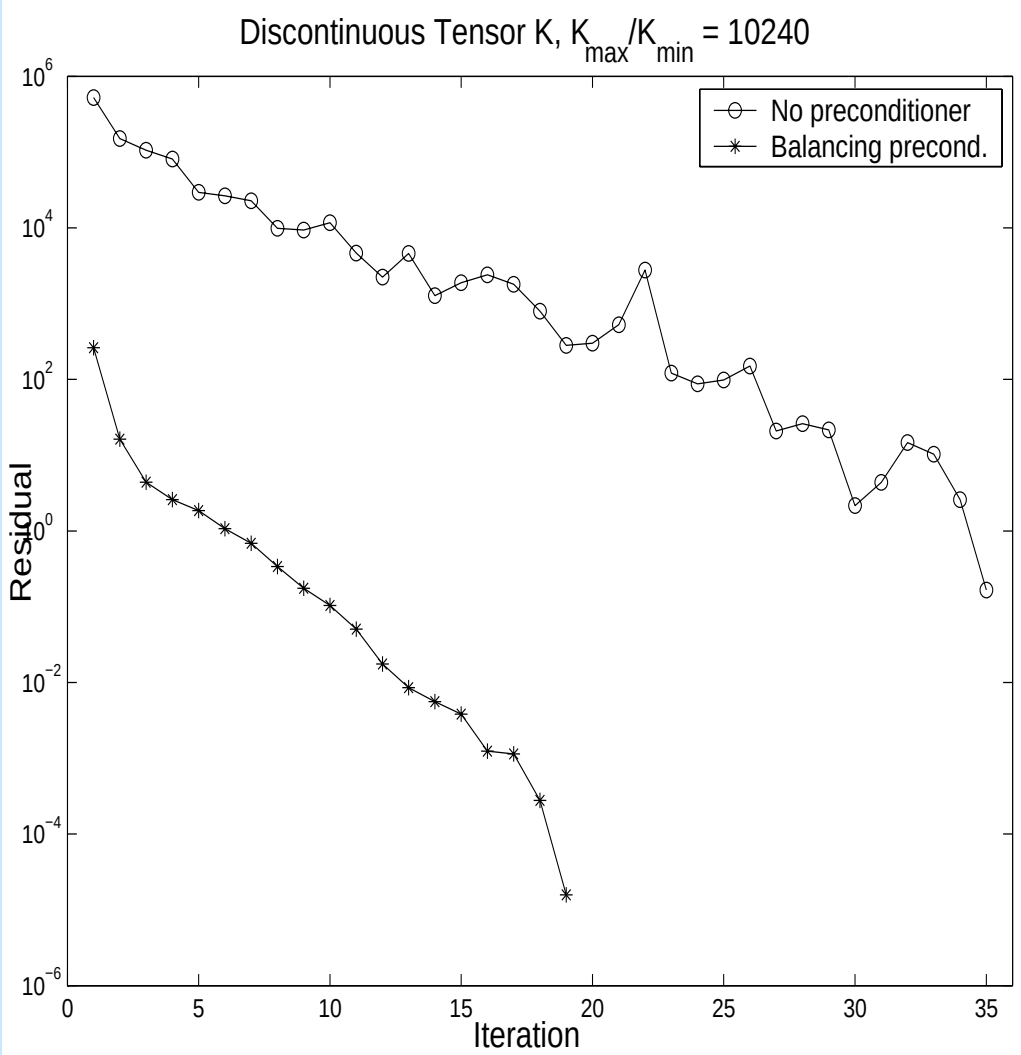




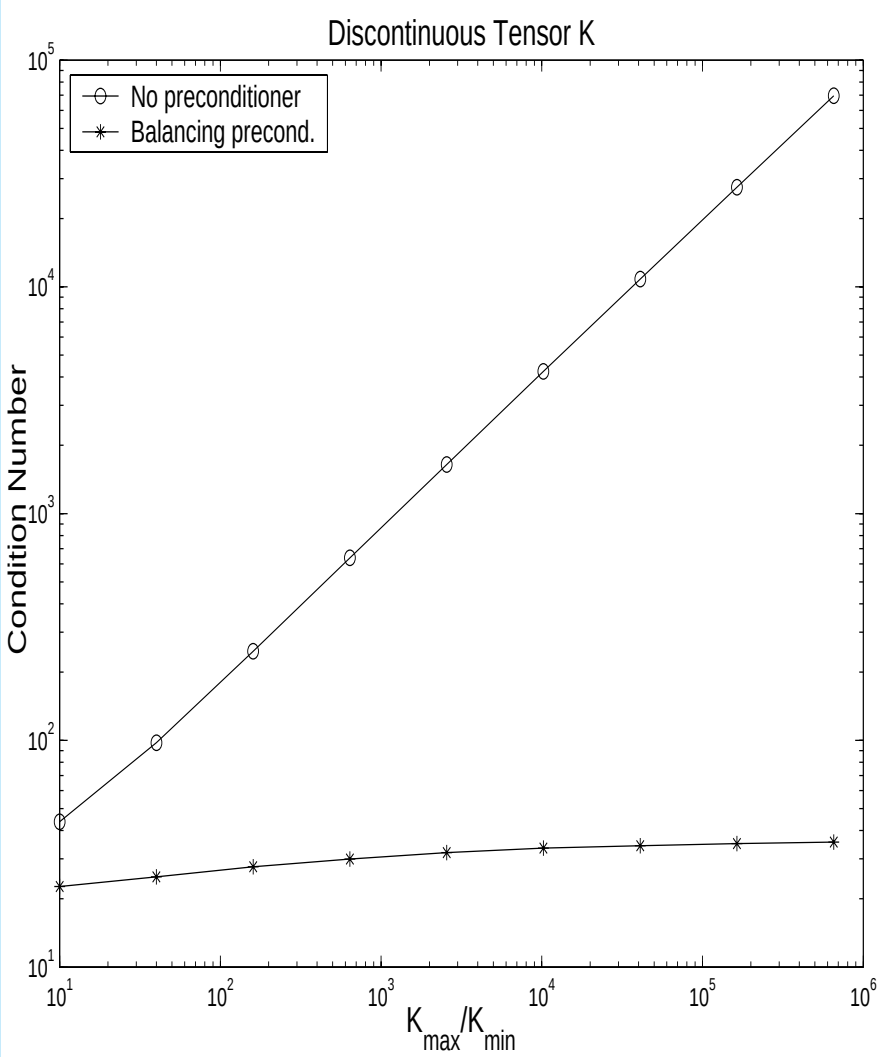


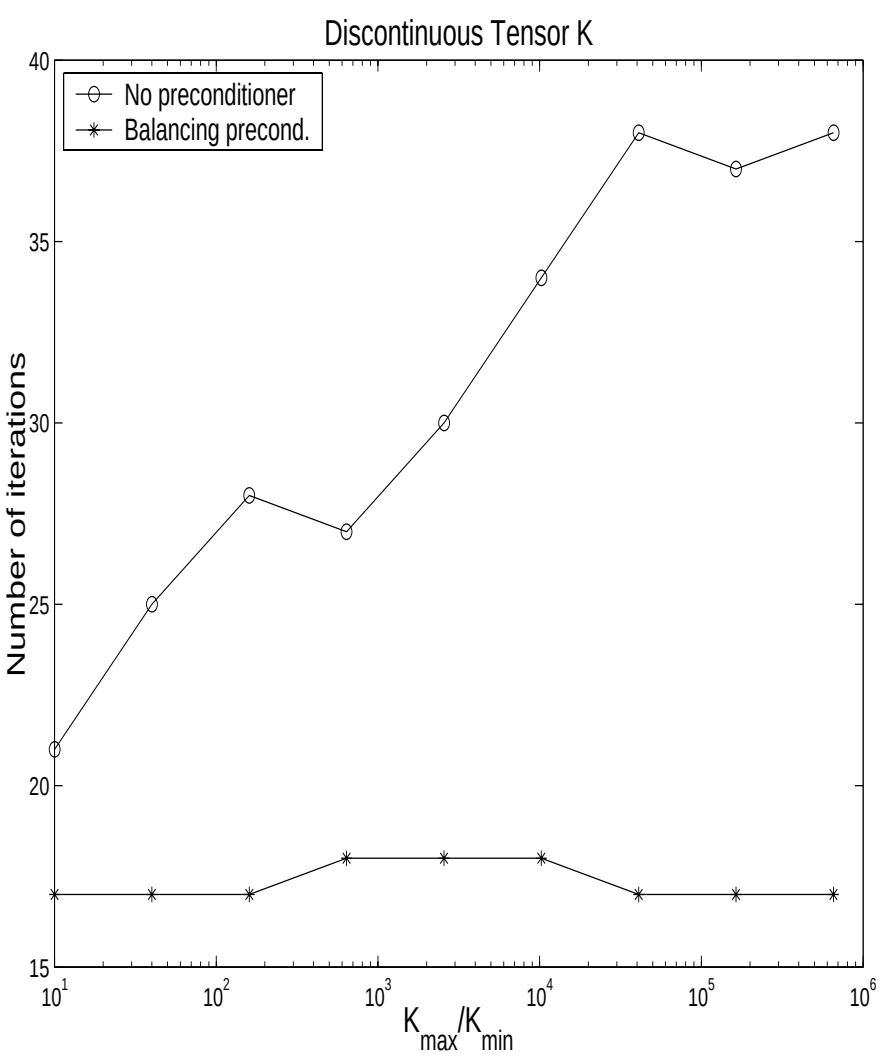
CG convergence with discontinuous coefficients





Dependence of CG convergence on jump in coefficients





Two-phase immiscible flow model

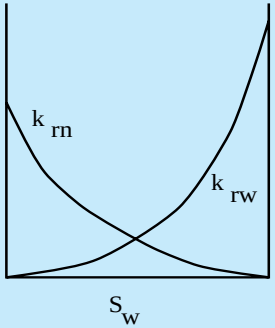
$$\mathbf{U}_\alpha = -\frac{k_\alpha(S_\alpha)K}{\mu_\alpha}\rho_\alpha(\nabla P_\alpha - \rho_\alpha g \nabla D) \qquad \text{(Darcy's law)}$$

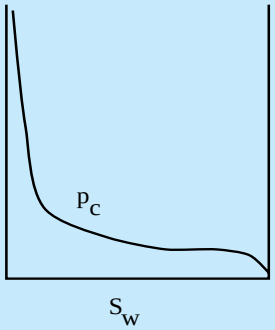
$$\frac{\partial(\phi\rho_\alpha S_\alpha)}{\partial t} + \nabla\cdot\mathbf{U}_\alpha = q_\alpha \qquad \text{(conservation of mass)}$$

$$S_w + S_n = 1, \qquad p_c(S_w) = P_n - P_w$$

$\alpha = w$ (wetting phase), n (non-wetting phase)

S_α - phase saturation	$N_\alpha = \rho_\alpha S_\alpha$ - concentration
ρ_α - phase density	P_α - phase pressure
ϕ - porosity	q_α - source term
K - permeability tensor	$\mathbf{U}_\alpha$ - Darcy velocity
$k_\alpha(S_\alpha)$ - phase rel. perm.	μ_α - phase viscosity
g - gravitational constant	D - depth





Multiblock formulation for two-phase immiscible flow model

On each subdomain Ω_i:

$$\mathbf{U}_\alpha = -\frac{k_\alpha(S_\alpha)K}{\mu_\alpha}\rho_\alpha(\nabla P_\alpha - \rho_\alpha g \nabla D) \qquad \text{(Darcy's law)}$$

$$\frac{\partial(\phi\rho_\alpha S_\alpha)}{\partial t} + \nabla\cdot\mathbf{U}_\alpha = q_\alpha \qquad \text{(conservation of mass)}$$

On each interface Γ_{ij}:

$$P_\alpha|_{\Omega_i} = P_\alpha|_{\Omega_j}, \qquad [\mathbf{U}_\alpha\cdot\nu]_{ij} = 0.$$

On each Ω_i and Γ_{ij}:

$$S_w + S_n = 1, \qquad p_c(S_w) = P_n - P_w.$$

Parallel domain decomposition - interface formulation

Fully implicit time discretization $\Rightarrow$
need to solve a nonlinear system on each time step.

Define interface operator $B : M \to M$ for $\lambda = (P_n^M, P_w^M) \in M$

$$B(\lambda) = ([\mathbf{U}_n^M(\lambda)], [\mathbf{U}_w^M(\lambda)]),$$

where $\mathbf{U}_\alpha^M(\lambda)$ is the mortar projection of the solution $\mathbf{U}_\alpha(\lambda) \cdot \nu$
to subdomain problems with Dirichlet boundary data λ.

The original problem is equivalent to solving for $\lambda \in M$ such that

$$B(\lambda) = 0.$$

Solution of the interface problem

The interface problem $B(\lambda) = 0$ is solved by inexact Newton method

$$\lambda_{k+1} = \lambda_k + s_k$$

using forward difference GMRES to approximate the Newton step

$$B'(\lambda_k)s_k = -B(\lambda_k).$$

Each GMRES iteration requires forward difference approximation of $B'(\lambda)s$:

$$D_\delta B(\lambda : s) = \frac{B(\lambda + \delta s) - B(\lambda)}{\delta}$$

Cost of a function evaluation

Algorithm for evaluation of $B(P_n^M, P_w^M)$.

1. Project (orthogonally) mortar data onto the subdomain grids

$$P_n^M \xrightarrow{Q_i} \bar{P}_{n,i}, \quad P_w^M \xrightarrow{Q_i} \bar{P}_{w,i}$$

2. Solve in parallel subdomain problems with boundary conditions $\bar{P}_{n,i}$, $\bar{P}_{w,i}$. Compute $\mathbf{U}_{n,i}$, $\mathbf{U}_{w,i}$ on each Ω_i .

3. Project boundary fluxes back to the mortar space

$$\mathbf{U}_{\alpha,i} \cdot \nu_i \xrightarrow{Q_i^T} \mathbf{U}_{\alpha,i}^M$$

4. Compute flux jump in the mortar space

$$[\mathbf{U}_\alpha^M]_{ij} = \mathbf{U}_{\alpha,i}^M + \mathbf{U}_{\alpha,j}^M$$

Interface GMRES preconditioner

$D_\delta B$: Pressure $\times$ Pressure $\rightarrow$ Flux $\times$ Flux
GMRES builds an orthonormal basis for the Krylov space.
Unpreconditioned GMRES step:

$$v_{k+1} = D_\delta B(\lambda : v_k)$$

$v_k \in$ Pressure $\times$ Pressure; $\quad v_{k+1} \in$ Flux $\times$ Flux
v_k and v_{k+1} live in different physical spaces!

Preconditioner: $M \sim (D_\delta B(\lambda))^{-1}$
M: Flux $\times$ Flux $\rightarrow$ Pressure $\times$ Pressure
Preconditioned GMRES step:

$$v_{k+1} = M D_\delta B(\lambda : v_k)$$

$v_{k+1} \in$ Pressure $\times$ Pressure
v_k and v_{k+1} live in the same physical space.
The preconditioned GMRES iterates in the Pressure $\times$ Pressure space.

Neumann-Neumann preconditioner for $D_\delta B$

$$D_\delta B = \sum_i D_\delta B_i$$

$D_\delta B$: Dirichlet to Neumann operator (Poincare-Steklov)

Preconditioner:

$$M = \sum_i \widehat{D_\delta B_i}^{-1},$$

where $\widehat{D_\delta B_i}^{-1}$ is an approximation to $D_\delta B_i^{-1}$.

M : Neumann to Dirichlet operator

Approximation to the Pressure $\rightarrow$ Flux interface operator

On Γ_{12}, $B = B_1 + B_2$,
B_i : Pressure $\rightarrow$ Flux is based on Darcy's Law

$$\mathbf{U} \cdot \nu_i = -(K \nabla P) \cdot \nu_i$$

Approximate B_i by a finite difference on a single-cell layer along the interface:

$$\hat{B}_i(\bar{P}) = K_i \frac{P_i - \bar{P}}{h_i/2}$$

Leading to an approximation to $D_\delta B(\bar{P} : s)$

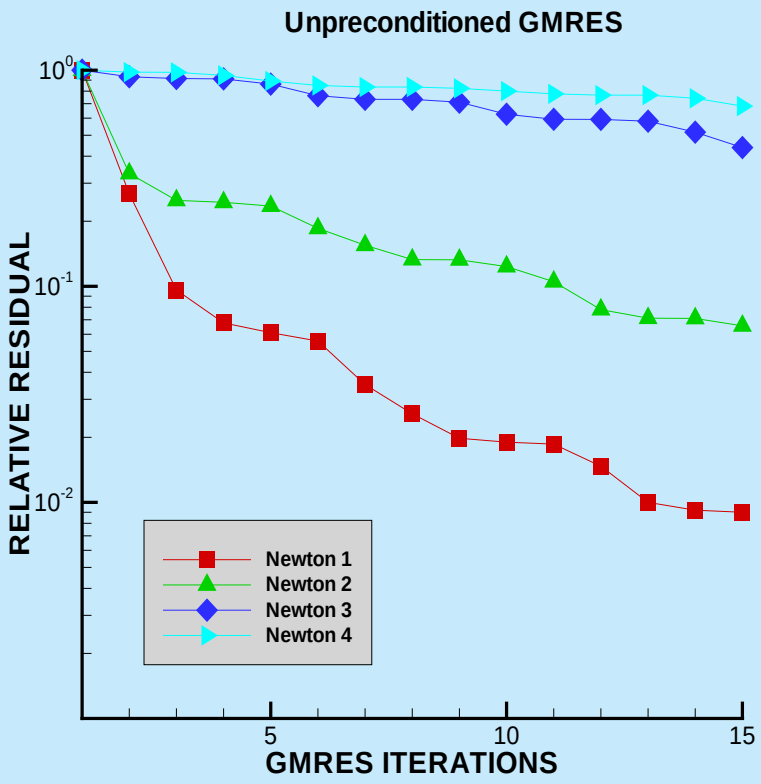
$$\widehat{D_\delta B}(\bar{P} : s) = \frac{\hat{B}(\bar{P} + \delta s) - \hat{B}(\bar{P})}{\delta} = -2(\frac{K_1}{h_1} + \frac{K_2}{h_2})s$$

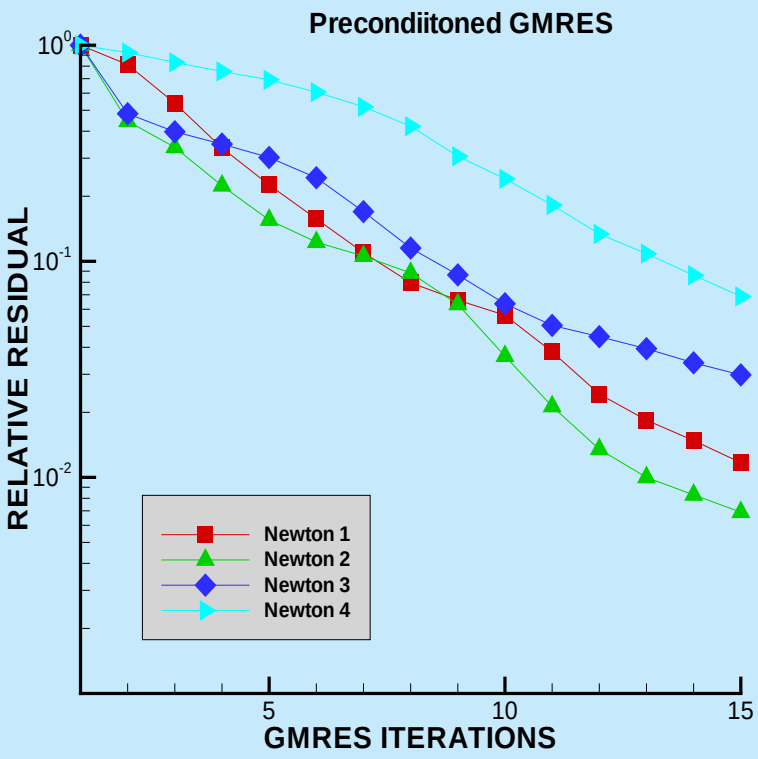
Preconditioner: given flux v

$$Mv = -\frac{1}{2(\frac{K_1}{h_1} + \frac{K_2}{h_2})} v$$

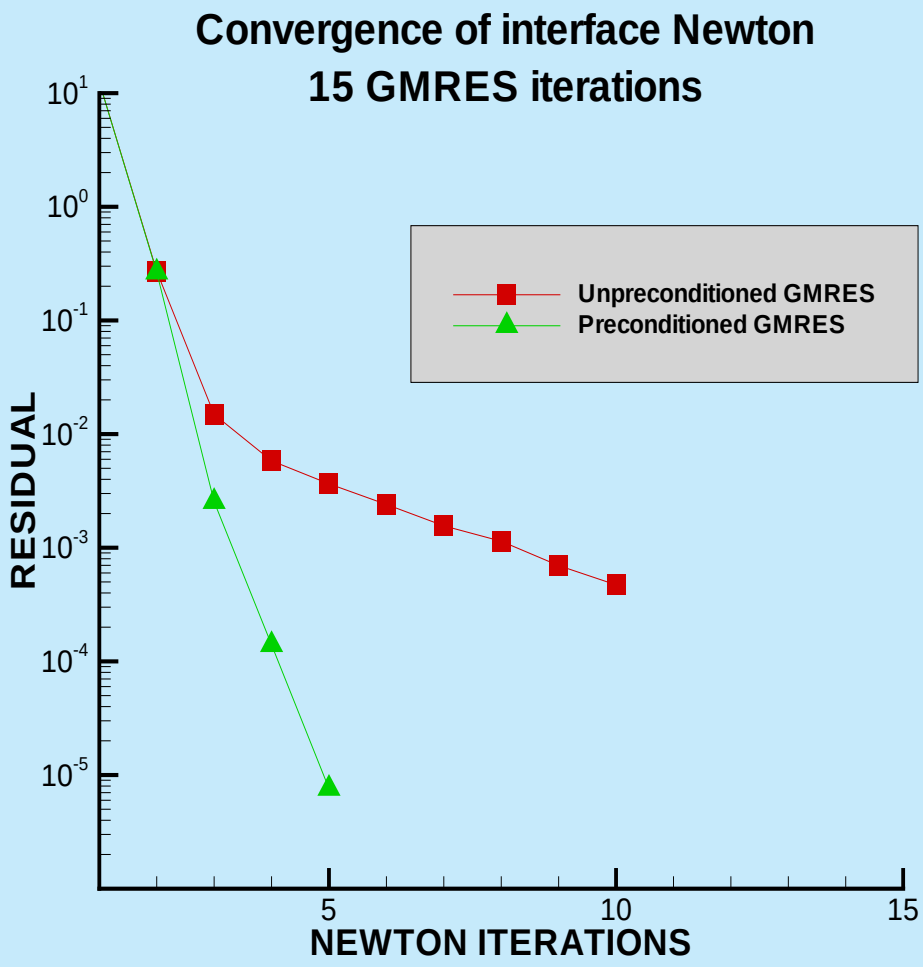
Interface GMRES convergence

Using the CSM simulator IPARS.





Effect of preconditioner on interface Newton convergence



Conclusions

- Multiblock formulation on non-matching grids
- Domain decomposition - interface problem
- Balancing preconditioner for single-phase flow
  - Consistent Neumann solves
  - Coarse solve provides global exchange of information
  - Quasi-optimal condition number independent of the jump in coefficients
- Newton-GMRES with Neumann-Neumann preconditioner for multiphase flow

Current work

- Balancing for multiphase flow
- Direct linearization of the global system
- IMPES-type formulation